

The Integral Test

Thm If $a_n > 0$, $a_n = f(n)$ where $f(x)$ is positive, continuous and decreasing for all $x \geq N$, $N \in \mathbb{N}$ (*)

Then $\sum_{n=N}^{\infty} a_n < \infty \iff \int_N^{\infty} f(x) dx < \infty$

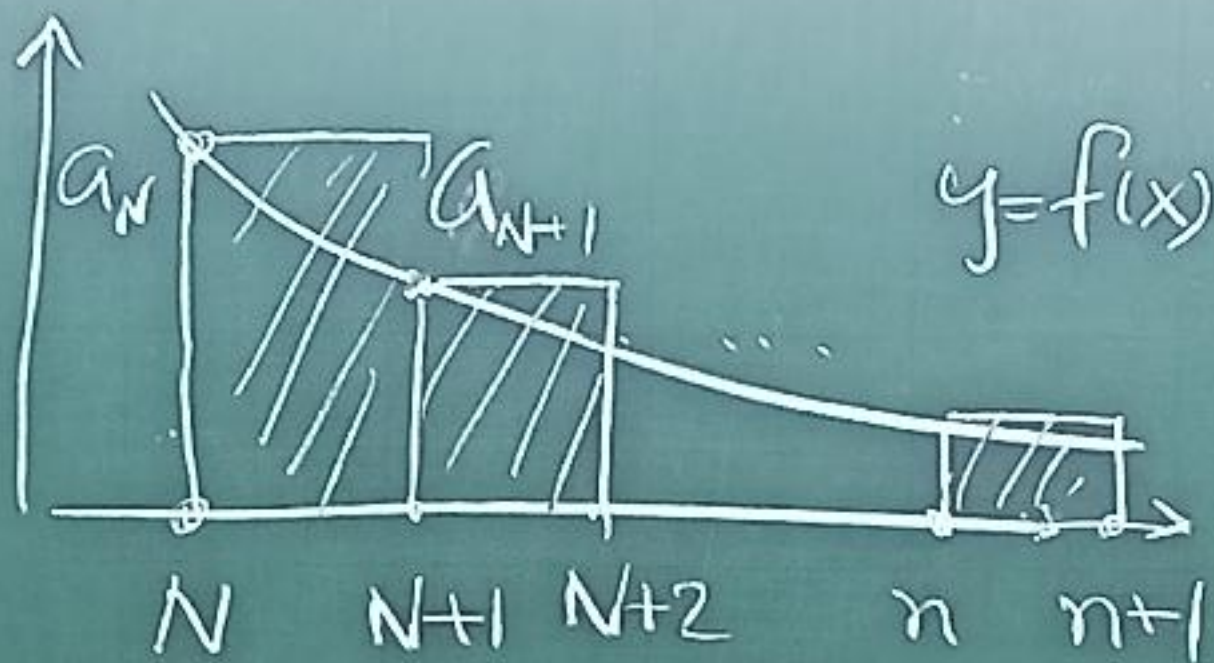
($a_n > 0$, $\sum_{n=N}^{\infty} a_n < \infty \iff \sum_{n=N}^{\infty} a_n$ conv. *)

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges?

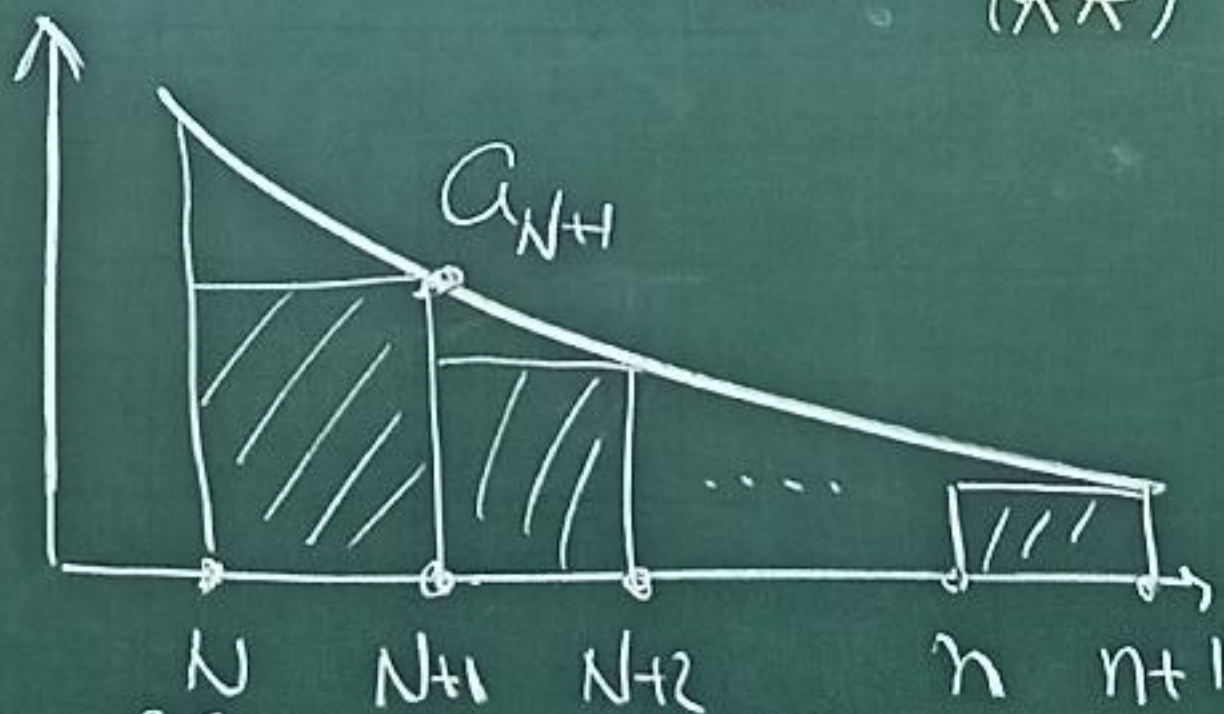
Sol $p \leq 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^p} \neq 0$, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$

$p > 0 \Rightarrow$ Compare with $\int_1^{\infty} \frac{1}{x^p} dx$ $\begin{cases} < \infty & p > 1 \\ = \infty & 0 < p \leq 1 \end{cases}$

pf.



$$\Rightarrow \sum_{n=N}^{\infty} a_n \geq \int_N^{\infty} f(x) dx \quad (**)$$



$$\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(x) dx \quad (***)$$

$$\int_N^{\infty} f(x) dx \leq \sum_{n=N}^{\infty} a_n \leq a_N + \int_N^{\infty} f(x) dx$$

(a_N is a fixed number)

$$\therefore \int_N^{\infty} f(x) dx < \infty \Leftrightarrow \sum_{n=N}^{\infty} a_n < \infty$$

R_M . If (*) holds, and

$$\sum_{n=N}^{\infty} a_n = L < \infty, \text{ then}$$

We can estimate R_M ($M > N$)

$$R_M = \sum_{n=N}^{\infty} a_n - \sum_{n=N}^M a_n \left(= \sum_{n=M+1}^{\infty} a_n \right)$$

$$\int_{M+1}^{\infty} f(x) dx \leq \sum_{n=M+1}^{\infty} a_n \leq \int_M^{\infty} f(x) dx$$

($N \rightarrow M+1$ in (**)) \downarrow ($N \rightarrow M$ in (***))

$$= R_M$$

$$\text{Eg 1. } \sum_{n=1}^{\infty} n^{-p} \begin{cases} = \infty, & 0 < p \leq 1 \\ < \infty, & p > 1 \end{cases}$$

$$\text{Eg 2. } \sum_{n=0}^{\infty} \frac{1}{1+n^2} < \infty$$

$$\text{Sol. } \int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx$$

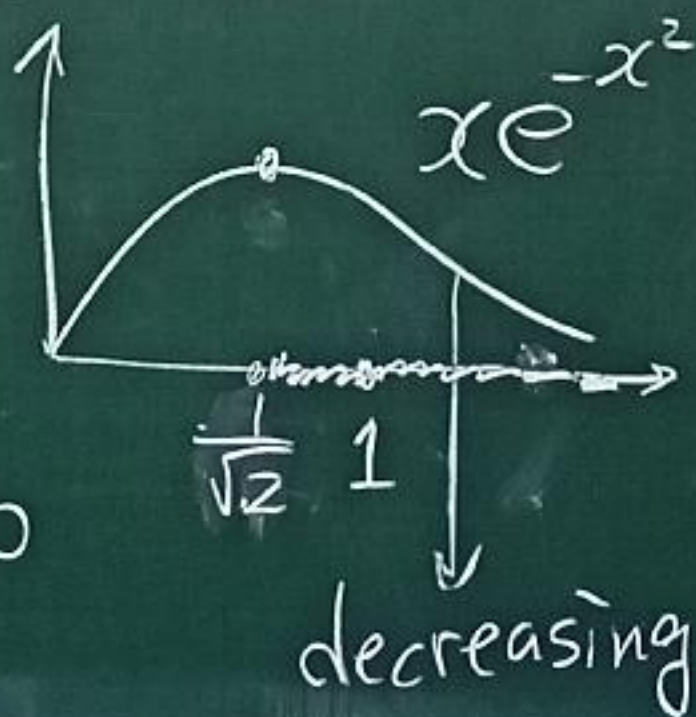
$$= \lim_{b \rightarrow \infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2} < \infty$$

$$\text{Eg 3. } \sum_{n=1}^{\infty} n e^{-n^2} < \infty$$

$$\text{Sol. } \int_0^{\infty} x e^{-x^2} dx$$

$$= \frac{1}{2} \left[-e^{-x^2} \right]_0^{\infty} < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} n e^{-n^2} < \infty$$



$$\text{Ex 4 } \sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} \leq \infty ?$$

$$\text{Sol: } \int_2^{\infty} \frac{1}{2^{\ln x}} dx = ?$$

$$y = \ln x, \quad x = e^y, \quad dx = e^y dy$$

$$= \int_{y=\ln 2}^{\infty} 2^{-y} e^y dy$$

$$= \int_{\ln 2}^{\infty} \left(\frac{e}{2}\right)^y dy = \infty$$

$\frac{e}{2} > 1$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} = \infty$$

Direct Comparison Test (Thm 10)

If $0 < d_n \leq a_n \leq c_n$

for all $n > N$, then

$$\textcircled{1} \sum_{n=1}^{\infty} c_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$$

$$\textcircled{2} \sum_{n=1}^{\infty} d_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$$

Ex 5 $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

Sol $\frac{5}{5n-1} > \frac{1}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n-1} = \infty$$

Limit Comparison Test (Thm 11)

If $a_n > 0, b_n > 0 \quad \forall n \geq N$

(i) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = C, 0 < C < \infty$

Then $\sum_{n=1}^{\infty} b_n < \infty \iff \sum_{n=1}^{\infty} a_n < \infty$
($= \infty$) ($= \infty$)

(ii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$

Then $\sum_{n=1}^{\infty} b_n < \infty \implies \sum_{n=1}^{\infty} a_n < \infty$

(iii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = +\infty$

Then $\sum_{n=1}^{\infty} b_n = \infty \implies \sum_{n=1}^{\infty} a_n = \infty$

$$\text{Ex 6 } \sum_{n=1}^{\infty} \frac{5}{5n+1}$$

$$\text{Sol } \frac{5}{5n+1} < \frac{1}{n}$$

\therefore Direct Comparison Test is inconclusive,

Limit Comparison Test

$$\lim_{n \rightarrow \infty} \frac{\frac{5}{5n+1}}{\frac{1}{n}} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n+1} = \infty$$

$$\text{Ex 7: } \sum_{n=0}^{\infty} \frac{1}{n!} < \infty \quad (0! \equiv 1)$$

$$\text{Sol: } 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$
$$< 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots < \infty$$

Eg 8. $\sum a_n$ $\sum b_n$ Thm Result

(a) $\sum_{n=1}^{\infty} \frac{2n}{(n+1)^2}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ $\frac{11}{(c=1)}$ (i) div.

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ $\sum_{n=1}^{\infty} \frac{1}{2^n}$ $\frac{11}{(c=1)}$ (i) conv.

(c) $\sum_{n=1}^{\infty} \frac{1+n \ln n}{n^2+5}$ $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\frac{11}{(c=1)}$ (i) div

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} < \infty ?$$

$$= \infty ?$$

Method 1. $\sum \frac{\ln n}{n} > \sum \frac{1}{n} = \infty$

Method 2. $\int_3^{\infty} \frac{\ln x}{x} dx = \infty \Rightarrow \sum_{n=3}^{\infty} \frac{\ln n}{n} = \infty$

($\frac{\ln x}{x}$ is decreasing for $x > e = 2.718...$)

$\sum a_n$	$\sum b_n$	Thm	Result
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(d) $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$	$\sum_{n=1}^{\infty} \frac{1}{n}$	11 (i) (c=1)	<u>div</u>
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(e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$	$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$	11 (i) (c=1)	<u>conv</u>
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(f) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$	$\sum_{n=1}^{\infty} \frac{n^{0.2}}{n^{3/2}}$	11 (ii) (c=0)	<u>conv</u>
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$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{(\ln n)^2}{n^{0.1}} \right)$$

$$= \left(\lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n^{0.1}} \right) \stackrel{\text{L'Hopital}}{=} \left(\lim_{n \rightarrow \infty} \frac{1}{0.1 n^{-0.9}} \right)^2$$

$$= 0$$