"Sequence" $= \{a_0, a_1, a_2, \cdots\}$

$$\left(= \{a_n\}_{n=0}^{\infty}\right)$$

How to define

$$\lim_{n\to\infty} a_n = L \ ?$$

Recall the definition of

$$\lim_{x\to\infty} f(x) = L$$

and Replace x by n
$f(x)$ by a_n

Def: $\lim_{n\to\infty} a_n = L$

if for every $\varepsilon > 0$, $(\varepsilon \in \mathbb{R})$

there exists a corresponding

integer N, such that

"$n > N \Rightarrow |a_n - L| < \varepsilon$"

(for all $n > N$, we have
$|a_n - L| < \varepsilon$)

Def: $\lim_{n\to\infty} a_n = \underline{\pm}\infty$

if for every $M \in \mathbb{R}$ $(\underline{m \in \mathbb{R}})$

there exists a corresponding integer N, such that

"$n > N \Longrightarrow a_n > M$ $(\underline{a_n < m})$"

(for all $n > N$, we have $a_n > M$ $(\underline{a_n < m})$)

Eg1. $\lim_{n\to\infty} \frac{1}{n} = 0$

Eg2. $\lim_{n\to\infty} n^2 = +\infty$

Eg3 $\{a_n\} = \{1, -1, 1, -1, \cdots\}$

$(a_n = (-1)^{n-1})$, $\lim_{n\to\infty} a_n$ diverges

Eg4 $\{a_n\} = \{1, 0, 2, 0, 3, 0, \cdots\}$

$\lim_{n\to\infty} a_n$ diverges.

How to find $\lim\limits_{n\to\infty} a_n$?

(I): Apply Sandwich Thm for sequences (Section 10.1 Theorem 2)

Eg5 $\lim\limits_{n\to\infty} \frac{\cos n}{n} = ?$

Sol: $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$

$\lim\limits_{n\to\infty} \frac{-1}{n} = 0 = \lim\limits_{n\to\infty} \frac{1}{n}$

$\xrightarrow{\text{Thm 2}} \lim\limits_{n\to\infty} \frac{\cos n}{n} = 0.$

(II) <u>Thm 4</u> Suppose

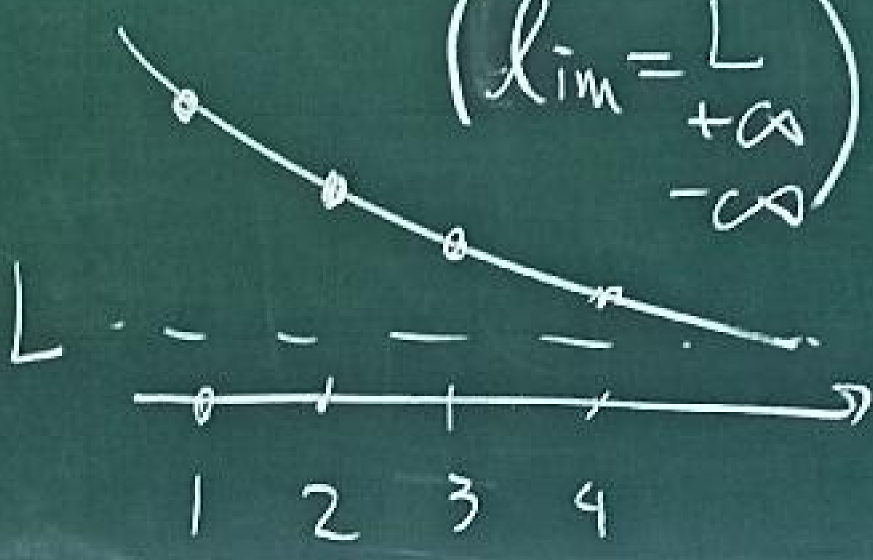
$f(x)$ is a function defined for all $x \geq n_0$ with $a_n = f(n)$

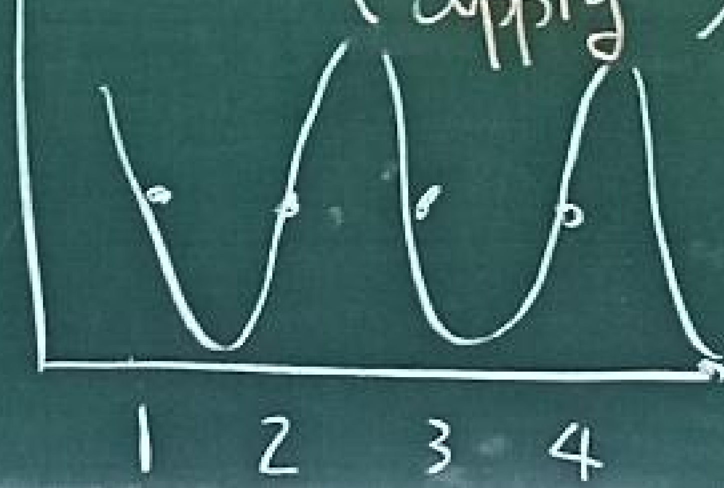
and $\lim\limits_{x\to\infty} f(x) = \begin{matrix} L \\ +\infty \\ -\infty \end{matrix}$

then $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x)$

$\left(\lim = \begin{matrix} L \\ +\infty \\ -\infty \end{matrix}\right)$

(Does not apply)





Eg 6 $\lim\limits_{n\to\infty} \frac{\ln n}{n} = ?$

Ans. Use Thm 4.

$$\lim_{x\to\infty} \frac{\ln x}{x} \left(= \frac{\text{"}\infty\text{"}}{\text{"}\infty\text{"}} \right)$$

$(x \in \mathbb{R},\ x > 0)$

l'Hôpital Rule

$$= \lim_{x\to\infty} \frac{(\ln x)'}{x'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 \quad \therefore \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Eg 7 $\lim_{n\to\infty} n^{\frac{1}{n}} = ?$

$$= \lim_{n\to\infty} \left(e^{\ln n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} e^{\frac{\ln n}{n}}$$

$$\overset{\text{Thm 3}}{=} e^{\left(\lim_{n\to\infty} \frac{\ln n}{n}\right)}$$

$$= e^{0} = 1.$$

Since $x \xrightarrow{f} e^{x}$ is continuous

$\therefore \lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right)$

Eg 8. $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n = ?$

Here $x \in \mathbb{R}$ is fixed.

<u>Sol</u>: "1^{∞}"

$$= \lim_{n\to\infty}\left(e^{\ln\left(1+\frac{x}{n}\right)}\right)^n$$

$$= \lim_{n\to\infty} e^{\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}}$$

$$= e^{\lim\limits_{n\to\infty}\left(\frac{\ln\left(1+\frac{x}{n}\right)}{n}\right)}$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{d}{dn}\left(\ln\left(1+\frac{x}{n}\right)\right)}{\frac{d}{dn}\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\left(-xn^{-2}\right)}{-n^{-2}}$$

$$= x$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$$

Eg 9 $\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n$

<u>Sol</u> $= \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^n$

$= \lim_{n\to\infty}\left(\left(1+\frac{2}{n-1}\right)^{n-1}\cdot\left(1+\frac{2}{n-1}\right)\right)$

$= \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^{n-1}\cdot\lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)$

$= e^2\cdot 1 = e^2$

Infinite Series

Def: $\sum_{n=1}^{\infty} a_n = \underset{(\pm\infty)}{L}$

if $\lim_{n\to\infty} S_n = \underset{(\pm\infty)}{L}$

where $S_n = \sum_{k=1}^{n} a_k$

$\left(= a_1 + a_2 + \cdots + a_n \right)$

i.e. $S_1, S_2, \cdots S_n,$

is the sequence of

partial sum of $\{a_k\}$

Examples of convergent Series

* Geometric Series

$$\sum_{n=0}^{\infty} ar^n = a + ar + \cdots + ar^n + \cdots$$

$$S_n = \sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} a\frac{1-r^{n+1}}{1-r}$$

$$= \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{div.} & \text{if } |r|\geq 1 \end{cases}$$

(裂項和)

* Telescoping Sum

Eg 10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Sol $S_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$

$= \displaystyle\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)$

$+ \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)$

$+ \cdots\cdots$

$+ \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$ $\therefore \displaystyle\sum_{n=1}^{\infty} = \lim_{n\to\infty} S_n = 1$

Thm (Section 10.2, Thm 7)

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \underset{(\not\Longleftarrow)}{\Longrightarrow} \lim_{n\to\infty} a_n = 0$$

$$\left(\sum_{n=1}^{\infty} a_n \text{ div} \Longleftarrow \lim_{n\to\infty} a_n \neq 0 \text{ or does not exist} \right)$$

Pf: Since $a_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} a_n = \left(\lim_{n\to\infty} S_n\right) - \left(\lim_{n\to\infty} S_{n-1}\right)$$

$$= L - L = 0$$

Eg of "$\not\Longleftarrow$" $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Examples of divergent Series

Eg 11 ($\lim_{n\to\infty} a_n \neq 0$ or D.N.E.)

$$\sum_{n=1}^{\infty} n^2, \ \sum_{n=1}^{\infty} (-1)^n, \ \sum_{n=1}^{\infty} \frac{n}{2n+5}$$

all diverge.

Eg 12 $\sum_{n=1}^{\infty} \frac{1}{n}$

Sol: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$

$> 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$

$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$

$= +\infty$