Remark on Definition of Differentiability

The following definitions are equivalent (i.e. the same, even though they look different)

Definition 1: $f(x,y)$ is differentiable at (x_0,y_0) if

$\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$ exist and

$$f(x,y)=L(x,y)+\varepsilon_1\cdot(x-x_0)+\varepsilon_2\cdot(y-y_0),\quad \lim_{(x,y)\to(x_0,y_0)}\varepsilon_1=\lim_{(x,y)\to(x_0,y_0)}\varepsilon_2=0,\tag{1}$$

(other equivalent definitions include:

$$f(x,y)=L(x,y)+\varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2},\qquad \lim_{(x,y)\to(x_0,y_0)}\varepsilon=0,\tag{2}$$

or

$$\Delta z=\partial_x f(x_0,y_0)\,\Delta x+\partial_y f(x_0,y_0)\,\Delta y+\varepsilon_1\cdot\Delta x+\varepsilon_2\cdot\Delta y\quad\text{(textbook version)}\tag{3}$$

or

$$\Delta z=\partial_x f(x_0,y_0)\,\Delta x+\partial_y f(x_0,y_0)\,\Delta y+\varepsilon\cdot\sqrt{(\Delta x)^2+(\Delta y)^2}\quad)\tag{4}$$

where

$$L(x,y)=f(x_0,y_0)+\partial_x f(x_0,y_0)\cdot(x-x_0)+\partial_y f(x_0,y_0)\cdot(y-y_0),\tag{5}$$

and

$$\Delta x=x-x_0,\quad \Delta y=y-y_0,\quad \Delta z=f(x,y)-f(x_0,y_0).\tag{6}$$

Remark 1:

In view of the identity (see homework 07, problem 2 for hint of proof)

$$\varepsilon_1\cdot(x-x_0)+\varepsilon_2\cdot(y-y_0)=\varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2},\tag{7}$$

it is easy to see that (1) and (2) are equivalent. Their equivalence to (3) and (4) are obvious.

The following Theorem shows that the assumption of existence of $\partial_x f(x_0, y_0)$ and $\partial_y f(x_0, y_0)$ is not essential in the definition of differentiability.
Theorem 1: If there exists $a,\ b \in \mathbb{R}$ such that the linear function

$$L(x, y) = f(x_0, y_0) + a \cdot (x - x_0) + b \cdot (y - y_0) \tag{8}$$

satisfies

$$f(x, y) = L(x, y) + \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0), \quad \lim_{(x,y)\to(x_0,y_0)} \varepsilon_1 = \lim_{(x,y)\to(x_0,y_0)} \varepsilon_2 = 0, \tag{9}$$

then $\partial_x f(x_0, y_0)$ and $\partial_y f(x_0, y_0)$ both exist and

$$\partial_x f(x_0, y_0) = a,\ \partial_y f(x_0, y_0) = b. \tag{10}$$

Proof:
We compute $\partial_x f(x_0, y_0) = \lim_{x\to x_0} \dfrac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$ directly. From (9) and (8), we have

$$\lim_{x\to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \lim_{x\to x_0} \frac{L(x, y_0) + \varepsilon_1 \cdot (x - x_0) - f(x_0, y_0)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{\big(f(x_0, y_0) + a \cdot (x - x_0)\big) + \varepsilon_1 \cdot (x - x_0) - f(x_0, y_0)}{x - x_0} = \lim_{x\to x_0} (a + \varepsilon_1) = a.$$

The proof for $\partial_y f(x_0, y_0)$ is similar.
In view of Theorem 1, we have a new and equivalent definition for differentiability:
Definition 1':

> $f(x, y)$ is differentiable at (x_0, y_0) if there exists $a,\ b \in \mathbb{R}$ such that the linear function
>
> $$L(x, y) = f(x_0, y_0) + a \cdot (x - x_0) + b \cdot (y - y_0) \tag{11}$$
>
> satisfies
>
> $$f(x, y) = L(x, y) + \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0), \quad \lim_{(x,y)\to(x_0,y_0)} \varepsilon_1 = \lim_{(x,y)\to(x_0,y_0)} \varepsilon_2 = 0. \tag{1}$$
>
> Here one can replace (1) by (2), (3), or (4).

Remark 3: It is straight forward to generalize Definition 1 and Definition 1' to higher dimensional case. For example, the 3D analogue of (2) reads

$$f(x, y, z) = L(x, y, z) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}, \quad \lim_{(x,y,z)\to(x_0,y_0,z_0)} \varepsilon = 0, \tag{12}$$

where

$$L(x, y, z) = f(x_0, y_0, z_0) + \partial_x f(x_0, y_0, z_0)\cdot(x - x_0) + \partial_y f(x_0, y_0, z_0)\cdot(y - y_0) + \partial_z f(x_0, y_0, z_0)\cdot(z - z_0). \tag{13}$$

3D analogues of (2), (3) and (4) are similar.

Question: How do we verify whether $f(x,y)$ is differentiable at (x_0,y_0)? For example, when $f(x,y)$ is of the form

$$f(x,y)=\begin{cases} \cdots & (x,y)\neq(0.0) \\ 0 & (x,y)=(0.0) \end{cases}.$$

Answer:
If you know the answer (differentiable or not):

Theorems to prove f is differentiable at (x_0,y_0):

Section 14.3, page 832, Theorem 3:

If f_x and f_y are continuous at (x_0,y_0), then f is differentiable at (x_0,y_0).

Note: INCONCLUSIVE if f_x or f_y is NOT continuous at (x_0,y_0).

Theorems to prove f is NOT differentiable at (x_0,y_0):

Section 14.3, page 832, Theorem 4:

If f is not continuous at (x_0,y_0), then it is not differentiable at (x_0,y_0).

Note: INCONCLUSIVE if f is continuous at (x_0,y_0).

Section 14.5, page 847, Theorem 9:

If $\left(\dfrac{df}{ds}\right)_{\boldsymbol{u},(x_0,y_0)}\neq\nabla f(x_0,y_0)\cdot\boldsymbol{u}$ for some direction $\boldsymbol{u}$, then f is not differentiable at (x_0,y_0).

If you are not sure whether f is differentiable at (x_0,y_0) or not:

Step 1: Find $\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$. If one of them does not exist, then $f(x,y)$ is NOT differentiable at (x_0,y_0).

Step 2: If $\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$ exist, we still need to check whether $L(x,y)$ given by (5) satisfies (1), (2). (3) or (4). In general, it is easier to check (2). In other words, to check whether the following is true:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0\,? \tag{14}$$

If (14) is true, then we can write

$$\varepsilon\overset{\text{def}}{=}\frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}},\qquad \lim_{(x,y)\to(x_0,y_0)}\varepsilon=0. \tag{15}$$

which is the same as (2). Thus we conclude that $f(x,y)$ is differentiable at (x_0,y_0) if and only if (14) holds. See Homework 09 problem 3 for some examples.