How to Check whether a vector field is conservative

1. Read and memorize the definition of "conservative vector field" on page 984.

2. $\boldsymbol{F}$ is conservative if and only if $\boldsymbol{F} = \nabla f$ for some potential function f (Theorem 1 + Theorem 2). If you read the proof of Theorem 2 carefully, you will see that the assumption "D is connected" is not necessary in Theorem 2. So being conservative is equivalent to having a potential function, no assumption on the domain needed.

3. Read the definitions of "connected" and "simply connected" on first paragraph of page 985 and the few figures there. They are very clearly exaplained.

4. The sphere $\{(x, y, z)\,|\,x^2 + y^2 + z^2 = 1\}$ is simply connected as illustrated in the figure below (from Wikipedia):

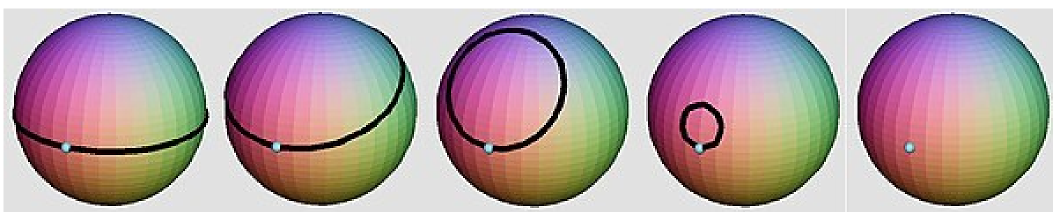

Figure 1: From left to right: a loop on the sphere shrinks (contracts) to a point while remaining on the sphere in the process.

5. Similarly, the ball with a hole at the center, $D = \{(x, y, z)\,|\,1^2 < x^2 + y^2 + z^2 < 2^2\}$, is simply connected as a loop in D can shrink to a point in D just like the previous example (the sphere). The key point here: a loop in D can avoid the center hole in the process of shrinking (contraction) in 3D. The radii 1 and 2 in D are arbitrary and not important.

6. Similarly, both $D = \{(x, y, z)\,|\,0 < x^2+y^2+z^2 < 1\}$ and $D = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ are simply connected for the same reason.

7. In contrast, a ring in the plane $\mathcal{R} = \{(x, y)\,|\,1^2 < x^2+y^2 < 2^2\}$ is not simply connected. This is very similar to the situation in Figure 16.22(c) on page 985. There if you try to shrink the loop C_1 to a point, you inevitably have to leave the region in the process. That is, the center whole in a 2D region cannot be avoided in the process of shrinking a loop around it. Similarly, the plane region $\mathcal{R} = \mathbb{R}^2 \setminus \{(0, 0)\}$ is not simply connected.

8. Unlike case 5 above, the domains $D = \{(x, y, z)\,|\,0 < x^2+y^2 < 1\}$ and $D = \mathbb{R}^3\setminus\{z\text{-axis}\}$ are not simply connected, since the "center whole" is no longer a point or a small ball, but a infinite long axis and therefore cannot be avoided in the process of shrinking a loop around it.

9. If $\boldsymbol{F}$ is conservative (i.e. $\boldsymbol{F} = \nabla f$), then $\boldsymbol{F}$ satisfies the component test (Equation (2) on page 988), since $f_{xy} = f_{yx}$, etc..

10. If $\boldsymbol{F}$ satisfies the component test and D is simply connected, then $\boldsymbol{F}$ is conservative ("Component Test for Conservative Fields", page 988). The proof requires results in Section 16.7 and is not given here, just memorize it for now.

11. If $\boldsymbol{F}$ satisfies the component test but D is NOT simply connected (that is, has a few unavoidable holes in D), then $\boldsymbol{F}$ may or may not be conservative. There are both conservative and non-conservative examples, see Eg2 and Eg3 on page 5-6 of Lecture 30.

12. In summary, we have the following methods to show $\boldsymbol{F}$ is or is not conservative:

    (a) If $\boldsymbol{F}$ does not satisfy the component test, then $\boldsymbol{F}$ is **not conservative**.

    (b) If $\boldsymbol{F}$ satisfies the component test, there are still two possibilities:

    (b1) If $\boldsymbol{F} = \nabla f$, then $\boldsymbol{F}$ is **conservative**.

    (b2) If $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} \neq 0$ for some closed loop C, then $\boldsymbol{F}$ is **not conservative**.

    To find out whether $\boldsymbol{F}$ is in case (b1) or (b2), we analyze as follows:

    i. If D is simply connected (i.e. no holes at all, or the holes are 'avoidable' as described above), then f must exist ("Component Test for Conservative Fields", page 988). Just proceed to find f so that $\boldsymbol{F} = \nabla f$ by direct integration as in Example 3, page 989. If you find f, then $\boldsymbol{F}$ is **conservative**.

    ii. If D seems to be not simply connected (has some holes, but not sure if they are "avoidable"), then find some closed loops C_1, C_2, $\cdots$, one around each hole.

    If $\displaystyle\oint_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r} \neq 0$ for some C_i, then $\boldsymbol{F}$ is **not conservative**.

    If $\displaystyle\oint_{C_i} \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for each of the C_i's, then it is most likely that $\boldsymbol{F}$ is **conservative**. Just proceed to find f so that $\boldsymbol{F} = \nabla f$ using the method as in Example 3, page 989. Notice that, as long as you can find such an f, you have shown that $\boldsymbol{F}$ is **conservative**. No need to worry about whether D is really simply connected or not.