Homework 08

1. Section 14.3: Problems 19, 21, 53, 60, 65, 67, 69, 72, 81, 91.

   Hint for problem 72:

   To compute
   $$f_{xy}(0,0) = \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{y - 0},$$
   we need $f_x(0,y)$ for $y \neq 0$ (direct differentiation) and
   $$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0}.$$
   The procedure for computing $f_{yx}(0,0)$ is similar.

2. Section 14.3: Further explanation on definition of differentiability on Lecture 14, page 6:

   Show that
   $$h(x,y) = \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0) \text{ with } \lim_{(x,y)\to(x_0,y_0)} (\varepsilon_1, \varepsilon_2) = (0,0) \tag{1}$$
   is the same as
   $$h(x,y) = \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2} \text{ with } \lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0. \tag{2}$$
   That is, if $h(x,y)$ is given by (1), then it can be rewritten (combined) as (2). Also, if $h(x,y)$ is given by (2), then it can be rewritten (split) as (1).

   Hint: Use
   $$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \Delta x + \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \Delta y$$
   or
   $$|\Delta x| \le \sqrt{(\Delta x)^2 + (\Delta y)^2}, \quad |\Delta y| \le \sqrt{(\Delta x)^2 + (\Delta y)^2},$$
   $$\sqrt{(\Delta x)^2 + (\Delta y)^2} \le |\Delta x| + |\Delta y|$$

   Remark: In terms of the notation 'O' ('Big O') and 'o' ('Big o') in Section 7.4 (page 464), the above statement can be recast as:
   $$o(1) \cdot \Delta x + o(1) \cdot \Delta y = o(1) \cdot \sqrt{(\Delta x)^2 + (\Delta y)^2}, \tag{3}$$
   or
   $$o(\Delta x) + o(\Delta y) = o(\sqrt{(\Delta x)^2 + (\Delta y)^2}), \tag{4}$$
   where the small $o(\cdot)$'s refer to $2D$ limit $\lim_{(\Delta x, \Delta y)\to(0,0)}$, same as $\lim_{(x,y)\to(x_0,y_0)}$.

   The statements (3) or (4) are more concise and widely used.

3. Section 14.4: Problems 1, 7, 10, 21, 24, 29, 31, 43, 51.