Brief solutions to selected problems in homework 15

1. Section 16.1: Solutions, common mistakes and corrections:

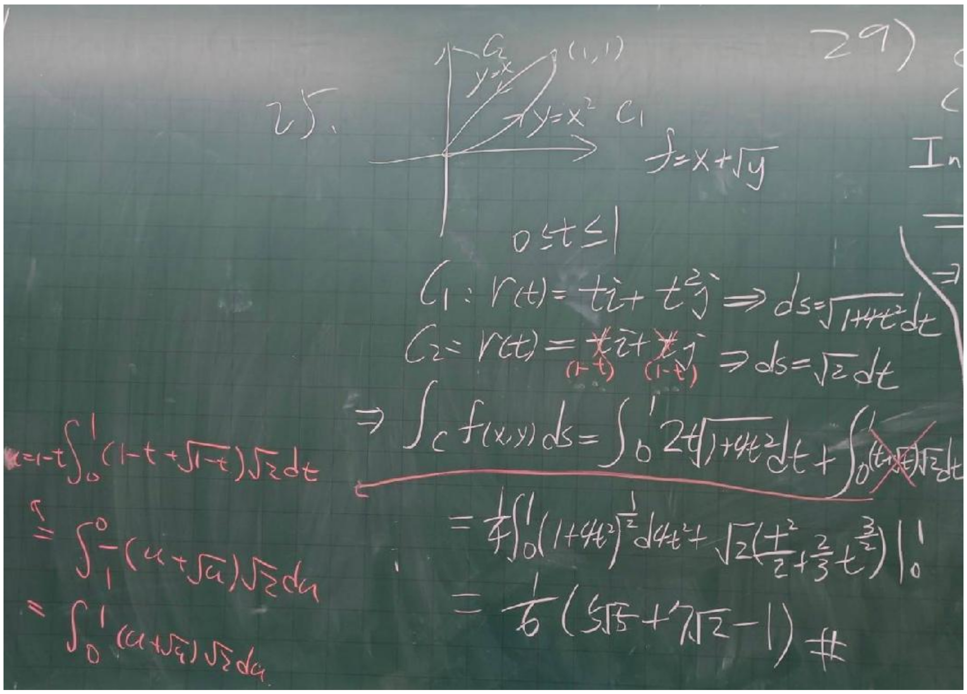

Figure 1: Solution to Section 16.1, problem 25

2. Section 16.3: Solutions, common mistakes and corrections:

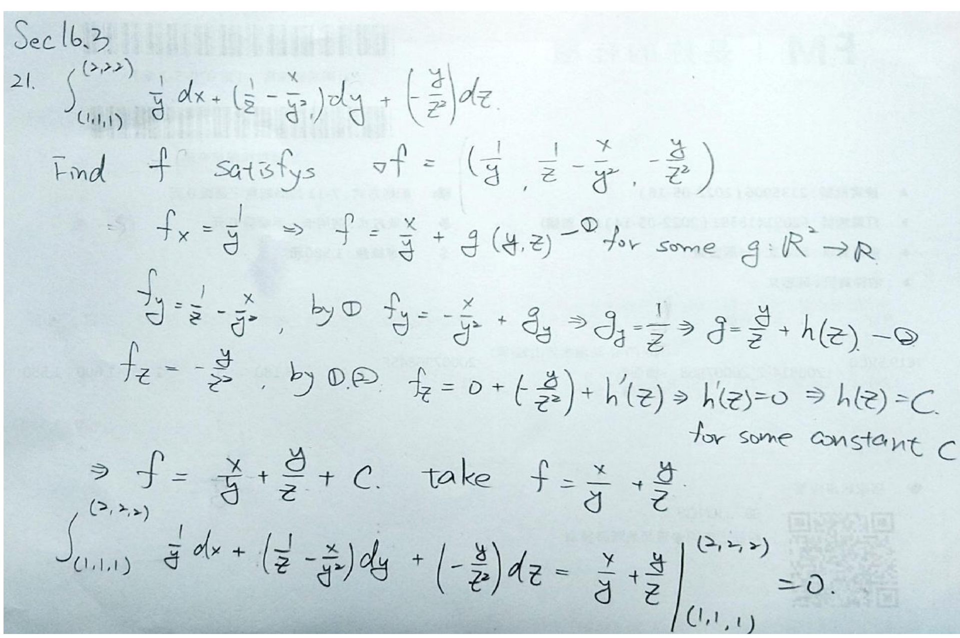

Figure 2: Solution to Section 16.3, problem 21

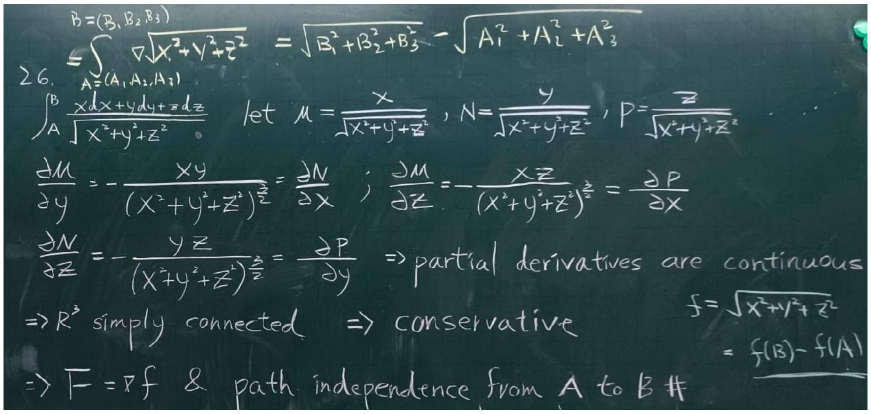

Figure 3: Solution to Section 16.3, problem 26

3. Homework 15, problem 4:

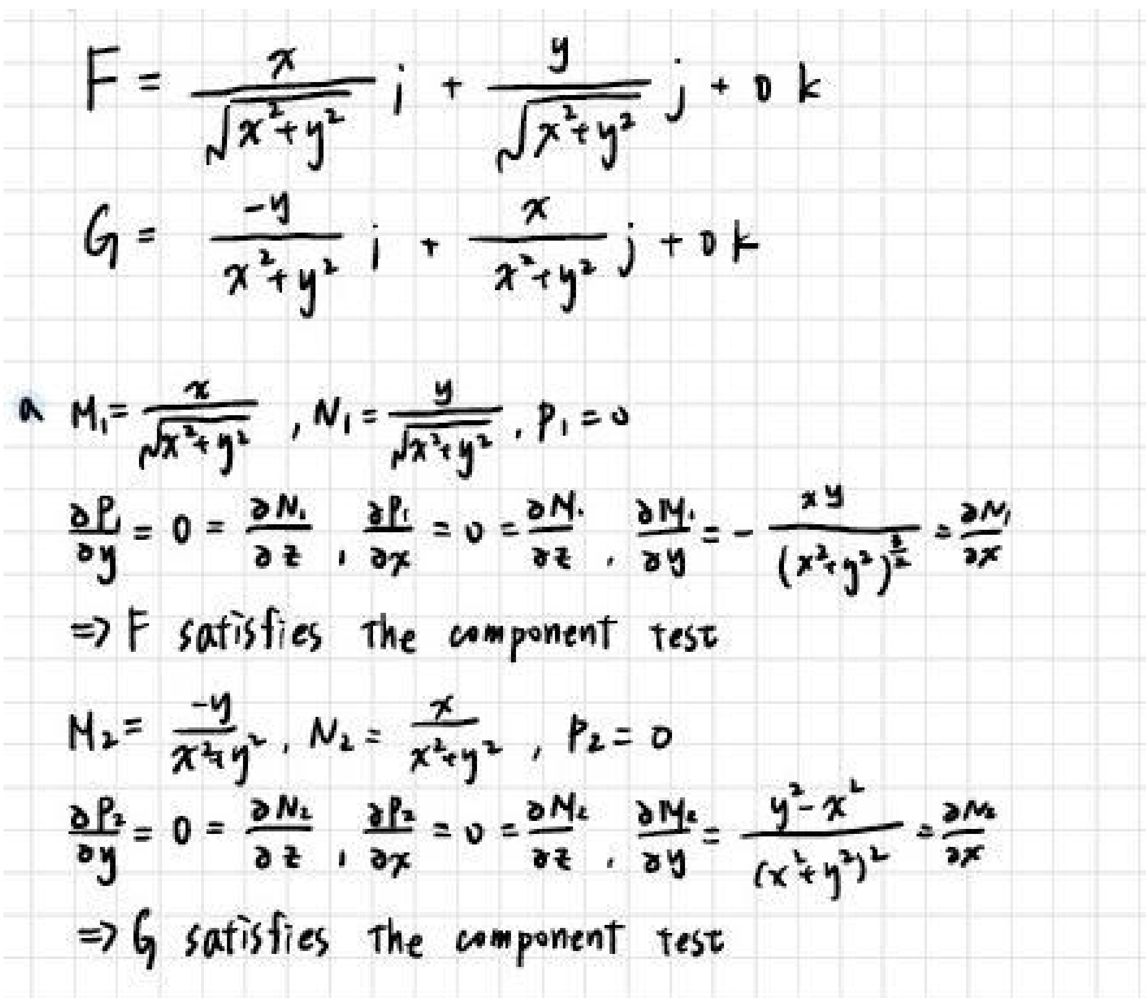

Figure 4: Solution to problem 4(a)

b $\nabla f = F$

$$\frac{\partial f}{\partial x} = M_1, \quad \frac{\partial f}{\partial y} = N_1, \quad \frac{\partial f}{\partial z} = P_1$$

$$f(x,y,z) = \sqrt{x^2+y^2} + g(y,z)$$

$$\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial g}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow \frac{\partial g}{\partial y} = 0$$

$$\Rightarrow f(x,y,z) = \sqrt{x^2+y^2} + h(z)$$

$$0 + \frac{\partial h}{\partial z} = 0 \Rightarrow \frac{\partial h}{\partial z} = 0, \quad h(z) = \cancel{z + C}\ C$$

$$\Rightarrow f(x,y,z) = \sqrt{x^2+y^2} \cancel{+z} + C$$

Figure 5: Solution to problem 4(b)

c $r(t) = (\cos t)\,i + (\sin t)\,j, \quad 0 \le t \le 2\pi$

$$G = \frac{-y}{x^2+y^2}\,i + \frac{x}{x^2+y^2}\,j$$

$$= \frac{-\sin t}{\sin^2 t + \cos^2 t}\,i + \frac{\cos t}{\sin^2 t + \cos^2 t}\,j$$

$$= (-\sin t)\,i + (\cos t)\,j$$

$$\frac{dr}{dt} = (-\sin t)\,i + (\cos t)\,j$$

$$\oint G \cdot dr = \oint_C G \cdot \frac{dr}{dt}\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= 2\pi \neq 0$$

$\because \oint G \cdot dr \neq 0$

$\therefore$ G isn't conservative by Thm 3.

Figure 6: Solution to problem 4(c)

Problem 4(d): It is easier to explain the idea if we restrict problem 4 in the plane:

Let $\boldsymbol{F} = \dfrac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{j}$ and $\boldsymbol{G} = \dfrac{-y}{x^2+y^2}\,\boldsymbol{i} + \dfrac{x}{x^2+y^2}\,\boldsymbol{j}$.

(a) Show that both $\boldsymbol{F}$ and $\boldsymbol{G}$ satisfy the component test.

(b) The natural domain of both $\boldsymbol{F}$ and $\boldsymbol{G}$ is $\{(x,y),\, x^2+y^2 \neq 0\}$ (that is where $\boldsymbol{F}$ and $\boldsymbol{G}$ are defined). Show that $\boldsymbol{F}$ is conservative in this domain by finding its potential function.

(c) Show that $\boldsymbol{G}$ is NOT conservative in this domain (see Example 5 on p990).

(d) If given another $\boldsymbol{H}$ satisfying the component test in this domain, how do you determine whether $\boldsymbol{H}$ is conservative?

Ans: It is clear that answers to (a), (b), (c) remain unchanged.

For (d): Suppose $\boldsymbol{H}$ satisfies the component test in $\{(x,y),\, x^2+y^2 \neq 0\}$. Let C be any simple closed curve, and $\mathcal{R}$ be the inside of C.

(i) If $(0,0) \notin \mathcal{R}$.

In this case, $\mathcal{R}$ is simply connected. We can apply the 2D version of 'Component Test for Conservative Field" statement on page 988, to conclude that ($\boldsymbol{H}$ is conservative, and therefore)

$$\oint_C \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0 \tag{1}$$

(ii) If $(0,0) \in \mathcal{R}$, we consider the domain $\mathcal{R}_\epsilon = \mathcal{R} \setminus \{x^2+y^2 \leq \epsilon^2\}$. Note that $\mathcal{R}_\epsilon$ is simply connected sine $(0,0) \notin \mathcal{R}_\epsilon$. Moreover, $\mathcal{R}_\epsilon = \mathcal{R}_{\epsilon,1} \cup \mathcal{R}_{\epsilon,2}$ as shown Figure 7.

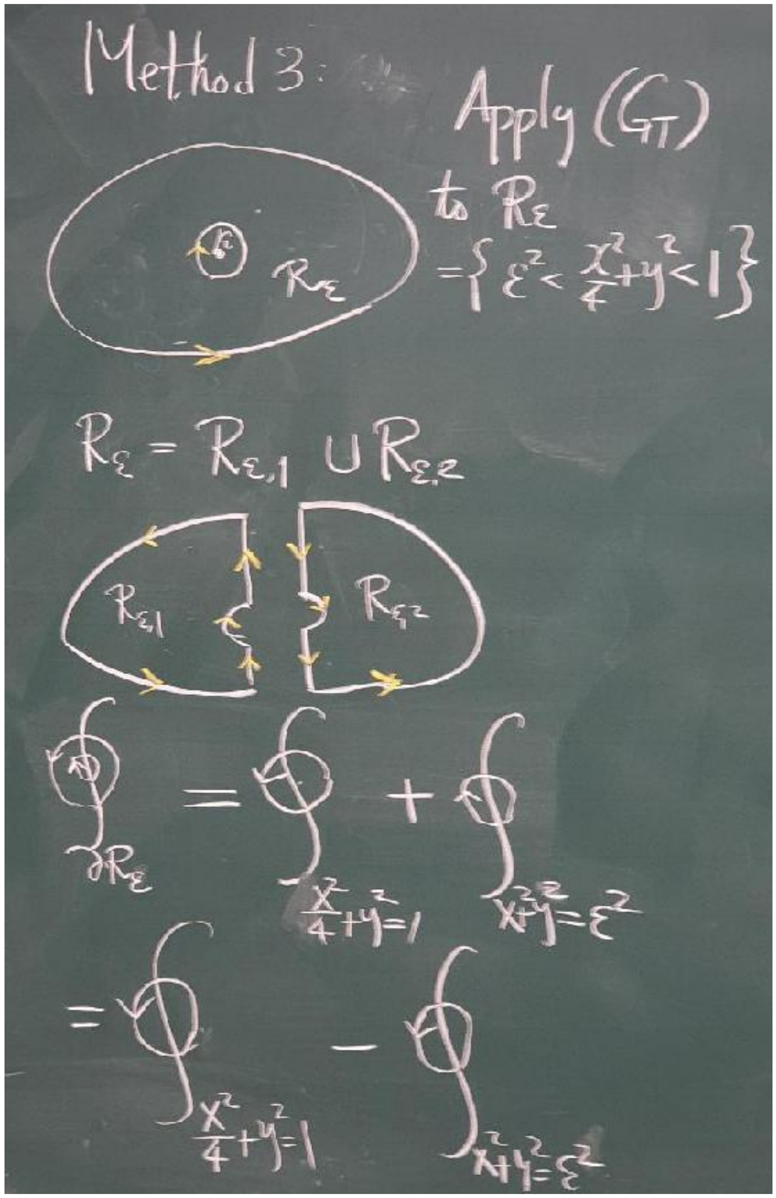

Figure 7:

Using the same argument as in case (a) above (replace $\mathcal{R}$ by $\mathcal{R}_{\epsilon,1}$ and $\mathcal{R}_{\epsilon,2}$, respectively), we have

$$\oint_{\partial\mathcal{R}_{\epsilon,1}} \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0, \qquad \oint_{\partial\mathcal{R}_{\epsilon,2}} \boldsymbol{H}\cdot\boldsymbol{T}\,ds = 0$$

where $\partial\mathcal{R}_{\epsilon,i}$ is the boundary of the region $\mathcal{R}_{\epsilon,i}$, $i = 1, 2$. As a result, we have

$$\oint_{C} \boldsymbol{H}\cdot\boldsymbol{T}\,ds = \oint_{C_\epsilon} \boldsymbol{H}\cdot\boldsymbol{T}\,ds \tag{2}$$

where $C_\epsilon = \{(x, y),\ x^2 + y^2 = \epsilon^2\}$. Moreover, it is clear that the line integral in (2) is independent of $\epsilon > 0$.

We conclude from the above analysis that,

(a) If $\oint\limits_{C_\epsilon} \boldsymbol{H} \cdot \boldsymbol{T}\, ds \neq 0$, then from Theorem 3 (loop property), $\boldsymbol{H}$ is not conservative.

(b) If $\oint\limits_{C_\epsilon} \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0$, then we conclude from (1), (2) that

$$\oint\limits_{C} \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0 \tag{3}$$

for every simple closed curve C.

If C is closed but not simple (i.e. C intersects itself), we can always decompose C into several simple closed curves (break up at the intersection points and reconnect), it follows that (3) remains valid even if C is not simple closed.

In summary, we have the following conclusion:

$$\boldsymbol{H} \text{ is conservative} \iff \oint\limits_{C} \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0 \text{ for any closed curve } C \iff \oint\limits_{C_\epsilon} \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0 \tag{4}$$

The conclusion (4) remains valid in 3D. The argument is similar, with the following replacement of key words:

2D: If C is simple closed and $(0,0) \notin \mathcal{R}$. (3D: If C does not circle around the z-axis).

2D: If C is simple closed and $(0,0) \in \mathcal{R}$. (3D: If C circles around the z-axis once).

2D: $C_\epsilon = \{(x,y),\ x^2+y^2=\epsilon^2\}$. (3D: $C_\epsilon = \{(x,y,z=0),\ x^2+y^2=\epsilon^2\}$).

2D: If C is not simple closed. (3D: If C circles around the z-axis more than once).

4. Problem 5:

Let $\vec{F} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x, y, z)$.

(a) What is the natural domain D_F of $\vec{F}$?

(b) Show that $\vec{F}$ satisfies component test in D_F.

(c) Is D_F simply connected?

(d) Is $\vec{F}$ conservative in this domain?

(a) $D_F = \{(x, y, z) \mid x^2+y^2+z^2 > 0\} = \mathbb{R}^3 \setminus \{(0,0,0)\}$

(b)

$$\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = -xy(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = -xz(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = -yz(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$\therefore$ $\vec{F}$ satisfies component test in D_F

(c)

D_F is simply connected

(d) By (b), $\vec{F}$ satisfies component test in D_F.

Also, D_F is simply connected.

$\therefore$ $\vec{F}$ is conservative in D_F

Method 2: By observation (or whatever methods), we know that $\boldsymbol{F} = \nabla\sqrt{x^2+y^2+z^2}$, therefore $\boldsymbol{F}$ is conservative.