Brief solutions to Quiz 8

May 23, 2023:

1. (33 pts) Evaluate $\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}\,dy\,dx$

   Ans:

$$=\int_0^{\pi}\int_0^{y}\frac{\sin y}{y}\,dx\,dy=\int_0^{\pi}(y-0)\frac{\sin y}{y}\,dy=\int_0^{\pi}\sin y\,dy=-\cos y\Big|_0^{\pi}=2$$

2. (34 pts) Change $\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y}\,dxdy$ into an equivalent polar integral and evaluate it.

   Ans:

   The region is enclosed by $x^2+y^2=4\,(r=2)$, $x=y\,(\theta=\frac{\pi}{4})$ and $y=2\,(r\sin\theta=2)$ in the first quadrant.

$$\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y}\,dxdy=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=2}^{\frac{2}{\sin\theta}}rdrd\theta=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{2}{\sin^2\theta}-2\right)d\theta$$

$$=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\left(2\csc^2\theta-2\right)d\theta=\left(-2\cot^2\theta-2\theta\right)\Big|_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}=2-\frac{\pi}{2}$$

3. (33 pts) Evaluate $\int_0^{\infty}\int_0^{\infty}\frac{1}{(1+x^2+y^2)^2}\,dy\,dx$

   Ans:

$$\int_0^{\frac{\pi}{2}}\int_0^{\infty}\frac{1}{(1+r^2)^2}\,rdr\,d\theta=\int_0^{\frac{\pi}{2}}\frac{1}{2}\int_{r=0}^{\infty}\frac{d(1+r^2)}{(1+r^2)^2}\,d\theta=\left(\int_0^{\frac{\pi}{2}}d\theta\right)\frac{-1}{2(1+r^2)}\Big|_{r=0}^{\infty}=\frac{\pi}{4}$$