Brief solutions to Quiz 7

May 02, 2023:

1. (a) Find the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$.
   Ans:
   $\nabla f(1, 1, 1) = (2, 4, 6)$.

   $$\text{Tangent plane: } \nabla f(1,1,1) \cdot (x-1, y-1, z-1) = 0, \text{ or } x + 2y + 3z = 6.$$

   (b) Find a tangent vector to the curve given by the intersection of the two surfaces $x^2 + y^2 + z^2 = 11$ and $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0$ at the point $(1, 1, 3)$.
   Ans:
   Let $f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x^3 + 3x^2y^2 + y^3 + 4xy - z^2$.
   $\nabla f(1, 1, 3) = (2, 2, 6)$. $\nabla g(1, 1, 3) = (13, 13, -6)$.
   A tangent vector: $(2, 2, 6) \times (13, 14, -6) = (-90, 90, 0)$, or $(-1, 1, 0)$.

2. Find the linearization of the $f(x, y) = x^2 - 3xy + 5$ near $(2, 1)$ and give an upper bound for the error $|E|$ of this approximation on the rectangle $|x - 2| \leq 0.1$, $|y - 1| \leq 0.1$.

   Ans:

   Linearization: $L(x, y) = f(2, 1) + f_x(2, 1)(x-2) + f_y(2, 1)(y-1) = 3 + (x-2) - 6(y-1)$.

   Error: $f_{xx} = 2$, $f_{xy} = -3$, $f_{yy} = 0$. $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} = 3$.

   $|f(x, y) - L(x, y)| \leq \frac{M}{2}\left(|\Delta x|^2 + |\Delta y|^2\right) \leq 0.06$.

3. Find all critical points of $f(x, y) = x^2 - xy + y^2 - 6x + 2$ and determine whether they are local min, local max or saddle points.

   Ans:

   $f_x = 2x - y - 6$, $f_y = -x + 2y$.

   Critical points: $(4, 2)$ only.

   $f_{xx}(4, 2) = 2$, $f_{xy}(4, 2) = -1$, $f_{yy}(4, 2) = 2$.

   $f_{xx} > 0$, $D = f_{xy}^2 - f_{xx}f_{yy} < 0$: local min.