Brief solutions to Quiz 6

Apr 25, 2023:

1. (34 pts) (**Average = 19.87 pts**)

   Let $F(x) = \int_0^{x^2} \cos(x^2 - t^2)\, dt$. Evaluate $F'(1)$. Give details.

   Ans:

   $$F'(x) = \cos(x^2 - (x^2)^2) \cdot 2x + \int_0^{x^2} -\sin(x^2 - t^2) \cdot 2x\, dt$$

   $$F'(1) = 2\cos(0) + \int_0^1 -2\sin(1 - t^2)\, dt = 2 - 2\int_0^1 \sin(1 - t^2)\, dt$$

2. (33 pts) (**Average = 24.03 pts**)

   Find the tangent line of $x^2 + \tan^2(y) = 2$ at $(x, y) = (1, \frac{\pi}{4})$.

   Ans:

   Let $f(x, y) = x^2 + \tan^2(y)$.

   $$f_x(x, y) = 2x, \quad f_x(x, y) = 2\tan(y)\sec^2(y),$$

   $$f_x(1, \frac{\pi}{4}) = 2, \quad f_x(1, \frac{\pi}{4}) = 2\tan(\frac{\pi}{4})\sec^2(\frac{\pi}{4}) = 4$$

   tangent line:

   $$\nabla f(1, \frac{\pi}{4}) \cdot (x - 1, y - \frac{\pi}{4}) = 0, \quad 2(x - 1) + 4(y - \frac{\pi}{4}) = 0.$$

3. (33 pts) (**Average = 28.00 pts**)

   Let $f(x, y) = x^2 - xy + 2y^2$. Find the direction $\boldsymbol{u}$ (a unit vector) for which the directional derivative $\left(\frac{df}{ds}\right)_{\boldsymbol{u},(1,1)}$ (that is, $D_{\boldsymbol{u}} f(1, 1)$) is largest, and find this directional derivative.

   Ans:

   $$f_x(x, y) = 2x - y, \quad f_x(x, y) = -x + 4y,$$

   $$f_x(1, 1) = 1, \quad f_x(1, 1) = 3,$$

   $$\boldsymbol{u} = \frac{\nabla f(1, 1)}{|\nabla f(1, 1)|} = \frac{(1, 3)}{\sqrt{10}}$$

   $$D_{\boldsymbol{u}} f(1, 1) = \nabla f(1, 1) \cdot \boldsymbol{u} = \sqrt{10}.$$