Brief solutions to Quiz 4

Mar 21, 2023: (**Average = 66.10 pts**)

Theorem A If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on $|x-a| < R$, $R > 0$, then $T_{f,a}(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = f(x)$ on $|x-a| < R$.

1. (20 pts + 20 pts) (**Average = 14.64 pts + 11.95 pts**)

   Give the power series representation (centered at $a = 0$) of $\frac{1}{1+x^2}$ on $|x| < 1$ (need not explain). Then use it to find the power series representation (centered at $a = 0$) of $\tan^{-1} x$ on $|x| < 1$ (explain).

   Ans:
   $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ on $|x| < 1$ (sum of geometric series).

   Apply Term by Term Integration Theorem to $\frac{1}{1+x^2} = (\tan^{-1} x)'$, we have

   $$\tan^{-1} x - \tan^{-1} 0 = \int_0^x \frac{1}{1+t^2} dt = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots) dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

   on $|x| < 1$.

2. (20 pts) (**Average = 11.65 pts**) Give an example of a power series that converges on $[0, 2]$ and diverges elsewhere. Explain all details.

   Ans: $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}$ will do. Since from ratio or root test, $\rho = 1$, therefore $R = 1$, the series converges on $|x-1| < 1$.

   On $|x-1| = 1$, the series converges absolutely in view of the p-series, $p = 2$. Therefore it converges on $|x-1| = 1$ by Absolute Convergence Test.

3. (20 pts + 20 pts) (**Average = 19.48 pts + 10.49 pts**) Suppose we know that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ on $x \in \mathbb{R}$.

   (a) Use it to find the power series representation of $\cos x$ (centered at $a = 0$). Explain.

   Ans:

   From Term by Term differentiation Theorem, we have

   $$\cos x = \frac{d}{dx} \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \text{ for all } x \in \mathbb{R}$$

(b) Let $f(x) = \dfrac{\sin x}{1-x}$ on $|x| < 1$. Use any method to find $T_{f,0}(x)$ up to and including the x^3 term.

Ans:

Since $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ (geometric series expansion) gives the power series representation of $\dfrac{1}{1-x}$ on $|x| < 1$. Therefore from The Series Multiplication Theorem for Power Series, $f(x) = (\sin x) \cdot \big(\dfrac{1}{1-x}\big)$ also has a power series representation on $|x| < 1$.

Method 1:

To find the power series representation of $f(x)$ on $|x| < 1$, we use the formula provided in The Series Multiplication Theorem for Power Series:

$$f(x) = \big(x - \frac{x^3}{3!} + \cdots\big) \cdot \big(1 + x + x^2 + x^3 + \cdots\big) = x + x^2 + \frac{5}{6}x^3 + \cdots$$

From Theorem A, this is the first few terms of $T_{f,0}(x)$.

Method 2:

Alternatively, one can find the first few terms of power series representation of $f(x)$ on $|x| < 1$ using the "Division of Power Series" formula in page 3 or page 4 of Lecture 07 note. From Theorem A, this is also the first few terms of $T_{f,0}(x)$.

Remark: More generally, if both $f(x)$ and $g(x)$ have power series representation on $|x - a| < R$ and $g(a) \neq 0$, then there exists a $\delta > 0$, such that $\dfrac{f(x)}{g(x)}$ has power series representation on $|x - a| < \delta$.

It is possible that $\delta < R$. For example, take $g(x) = 1 + x^2$. The power series representation of $g(x)$, centered at $a = 0$, is given by $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with $b_0 = 1$, $b_1 = 0$, $b_2 = 1$, $b_3 = b_4 = \cdots = 0$. So radius of convergence for $g(x)$ is $R = \infty$. On the other hand, radius of convergence for $\dfrac{1}{g(x)} = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ is $\delta = 1$.

The proof of this statement is beyond the scope of this course (key word: "Complex Analysis" course). So from now on, if you choose to use power series division (undetermined coefficients or long division) in the exams in this course, you may simply take the above statement for granted (i.e. power series representation for $\dfrac{f(x)}{g(x)}$ indeed exists on a small interval) and proceed with the calculation.

Method 3: (Not recommended for this problem since the calculation is complicated)

Evaluate $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ directly to get the first few terms of $T_{f,0}(x)$.