Brief solutions to Quiz 1

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1. (30 pts) Write down the definition of $\lim_{n\to\infty} a_n = \infty$ and use it prove $\lim_{n\to\infty}(n^3+1) = \infty$.

   Ans:

   Definition (**15 pts**): See page 589 of the textbook or page 6 of Lecture 01 note.

   Proof (**15 pts**): For any $M \in \mathbb{R}$, take $N \in \mathbb{N}$, $N > (M-1)^{\frac{1}{3}}$.

   Then for all $n > N$, we have

   $$a_n = n^3 + 1 > N^3 + 1 > M.$$

   This proves that $\lim_{n\to\infty} n^3 + 1 = \infty$.

2. (20 pts) Evaluate $\lim_{n\to\infty}\left(1 - \frac{2}{n}\right)^n$. Give details.

   Ans:

   Answer $= e^{-2}$. See page 9-10 of Lecture 02 for details.

3. (20 pts) Find all $x \in \mathbb{R}$ such that $\sum_{n=0}^{\infty}(\ln x)^n$ converges and find the corresponding sum.

   Ans:

   See page 2 of Homework 01 solution.

   Remark: By convention, $\sum_{n=0}^{\infty}(\ln x)^n$ exactly means $1 + \sum_{n=1}^{\infty}(\ln x)^n$ as a function of x. So you don't need to worry about 0^0 when you try to evaluate it at $x = 1$.

4. (30 pts) Write down the definition of $\sum_{n=1}^{\infty} a_n = L$ and use it to evaluate $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

   Ans:

   Definition (**15 pts**): See page 599 of the textbook, or page 12 of Lecture 02 note.

   Evaluation (**15 pts**): $\sum_{n=1}^{\infty}\frac{1}{n^2(n^2+1)}$ was a typo. My mistake. 15 pts free for everyone.

   From the identity $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we have

   $$s_k = \sum_{n=1}^{k} a_n = 1 - \frac{1}{k+1}.$$

Therefore

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = 1.$$