Brief solutions to Midterm 1 (**Average = pts**)

Mar 28, 2023.

1. (12 pts) True or False? Prove it if true and give an counter example if false.

   If $f(x,y)$ is differentiable at $(0,0)$, then $f(x,y)$ is continuous at $(0,0)$.

   Answer. True. Since f is differentiable at $(0,0)$, both $f_x(0,0)$ and $f_y(0,0)$ exist and

   $$f(x,y)-f(0,0)=f_x(0,0)x+f_y(0,0)y+\epsilon_1 x+\epsilon_2 y \text{ where } \epsilon_1,\epsilon_2\to 0 \text{ as } x,y\to 0$$

   $$\Rightarrow |f(x,y)-f(0,0)|\le |f_x(0,0)||x|+|f_y(0,0)||y|+|\epsilon_1||x|+|\epsilon_2||y|\to 0 \text{ as } (x,y)\to(0,0).$$

2. (12 pts) Evaluate

   $$\frac{d}{dy}\int_1^{2+y^2}\frac{\cos(xy)}{x}dx.$$

   Answer. Let $G(y,z)=\displaystyle\int_1^z \frac{\cos(xy)}{x}\,dx$. Then we need to compute

   $$(G(y,2+y^2))' = G_y(y,2+y^2)+G_z(y,2+y^2)\cdot 2y = -\int_1^{2+y^2}\sin(xy)\,dx+\frac{\cos((2+y^2)y)}{2+y^2}\cdot 2y$$

   $$=\frac{1}{y}[\cos((2+y^2)y)-\cos y]+\frac{\cos((2+y^2)y)}{2+y^2}\cdot 2y.$$

3. (12 pts) Find the equation of plane normal to the following curve at $(1,-1,1)$

   $$\begin{cases} x^2+2y^2+3z^2=6 \\ x+y+z=1 \end{cases}$$

   Answer. First compute the two gradients at $(1,-1,1)$

   $$(2x,4y,6z)_{(1,-1,1)}=(2,-4,6)$$

   and

   $$(1,1,1)|_{(1,-1,1)}=(1,1,1).$$

   The normal vector of the plane is parallel to the outer product of these two gradients:

   $$n=\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}=(5,-2,-3).$$

   Therefore, the equation of the plane is

   $$5(x-1)-2(y+1)-3(z-1)=0.$$

4. (12 pts) Show that, for all $a \in \mathbb{R}$, the point $(x_0, y_0) = (0, 0)$ is a critical point of the function

$$f_a(x, y) = (a - 2)(2x + y)^2 + (a + 1)(x - 2y)^2.$$

For what values of a does f_a have a local minimum, a local maximum or a saddle point at $(0, 0)$, respectively?

Answer.

Firstly, we compute the gradient of f_a at $(0, 0)$.

$$\nabla f_a(0,0) = (2(a-2)(2x+y)2+2(a+1)(x-2y),\ 2(a-2)(2x+y)+2(a+1)(x-2y)(-2))|_{(0,0)} = (0,0),$$

thus $(0, 0)$ is a critical point.

Since $(2x + y)^2 > 0$, $(x + 2y)^2 > 0$ for $(x, y) \neq (0, 0)$, we can determine the nature of the critical point by the signs of the coefficients $a - 2$ and $a + 1$.

If $a \geq 2$, $f_a(x, y) = (a - 2)(2x + y)^2 + (a + 1)(x - 2y)^2$ is the sum of a non-negative term and a positive term. Therefore f_a has a local the minimum at $(0, 0)$.

If $-1 < a < 2$, $f_a(x, y) = (a + 1)(x - 2y)^2 - (2 - a)(2x + y)^2$ is the difference of two positive terms. Therefore f_a has a saddle point at $(0, 0)$.

If $a \leq -1$, $f_a(x, y) = (a - 2)(2x + y)^2 + (a + 1)(x - 2y)^2$ is the sum of is the sum of a negative term and a non-positive term. Therefore f_a has a local the maximum at $(0, 0)$.

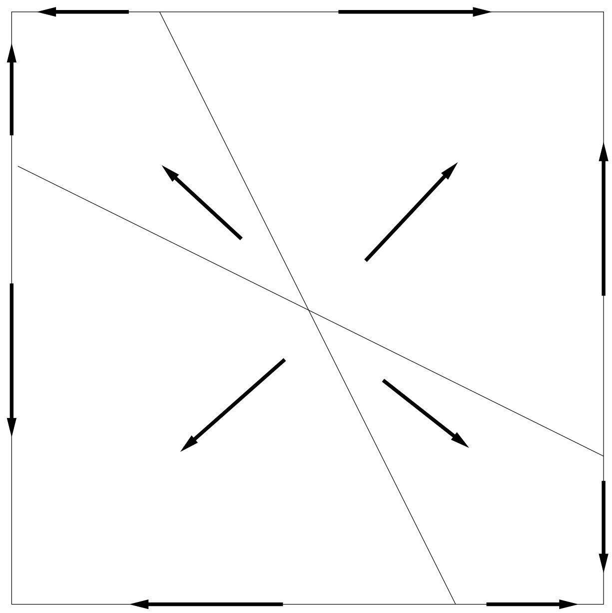

Figure 1: The gradient analysis for problem 5.

5. (12 pts) Find absolute maxima and minima of $f(x,y) = x^2+xy+y^2$ on the rectangular $-2 \le x \le 2$, $-2 \le y \le 2$.

   Answer.

   First find the gradient:
   $$\nabla f(x,y) = (2x+y, x+2y).$$
   It follows that
   $$f_x \gtrless 0 \text{ if } 2x+y \gtrless 0; \quad f_y \gtrless 0 \text{ if } x+2y \gtrless 0,$$
   Therefore one can plot the gradient vectors in the interior and the tangential component of the gradient vectors on the boundary as in Figure 1. From the plot it is easy to see that $f(0,0) = 0$ is indeed local and absolute minima. Moreover, the local maximum consists of the four corners. Upon comparing the values of f on the four corners, it follows that the absolute maxima is $f(2,2) = f(-2,-2) = 12$.

6. (12 pts) Give an example of a constraint optimization problem that, upon applying the method of Lagrangian multipliers, results in a system of 5 equations with 5 unknowns (and write down the 5 equations). Need not solve it.

   Answer. For example: Find extreme values of $f(x,y,z) = x^2+y^2+z^2$, subject to the constraints $g_1(x,y,z) = x^2+2y^2+3z^2-1 = 0$, and $g_2(x,y,z) = z = 0$.

$$\begin{cases} 2x = 2\lambda_1 x \\ 2y = 4\lambda_1 y \\ 2z = 6\lambda_1 z + \lambda_2 \\ x^2+2y^2+3z^2-1 = 0 \\ z = 0. \end{cases}$$

Example II:

Find extreme values of $f(x,y,z,w) = x^2+y^2+z^2+w^2$, subject to the constraint $g(x,y,z,w) = x^2+2y^2+3z^2+4w^2 = 5$.

$$\begin{cases} 2x = 2\lambda x \\ 2y = 4\lambda y \\ 2z = 6\lambda z \\ 2w = 8\lambda w \\ x^2+2y^2+3z^2+4w^2 = 5 \end{cases}$$

7. (12 pts) Use Taylor's formula to find the quadratic approximation of $f(x,y,z) = \dfrac{1}{1-x-y+z}$ near the origin.

Answer. Find all of the first and second derivatives first.

$$f_x(0,0,0) = f_y(0,0,0) = -f_z(0,0,0) = \left.\frac{1}{(1-x-y+z)^2}\right|_{(0,0,0)} = 1,$$

and

$$\begin{aligned} f_{xx}(0,0,0) = f_{yy}(0,0,0) = f_{xy}(0,0,0) = f_{zz}(0,0,0) = -f_{yz}(0,0,0) \\ = -f_{zx}(0,0,0) = \frac{2}{(1-x-y+z)^3} = 2. \end{aligned}$$

Therefore, the quadratic approximation is

$$\begin{aligned} Q(x,y,z) &= f(0,0,0) + f_x(0,0,0)x + f_y(0,0,0)y + f_z(0,0,0)z \\ &\quad + \frac{1}{2}\Big(f_{xx}(0,0,0)x^2 + f_{yy}(0,0,0)y^2 + f_{zz}(0,0,0)z^2 \\ &\quad + 2f_{xy}(0,0,0)xy + 2f_{yz}(0,0,0)yz + 2f_{zx}(0,0,0)zx \Big) \\ &= 1 + x + y - z + \left(x^2+y^2+z^2+2xy-2yz-2zx\right). \end{aligned}$$

Method II:

Let $w = x+y-z$ and consider Taylor's formula for $\dfrac{1}{1-w}$. Since the quadratic approximation of $\dfrac{1}{1-w}$ is $Q(w) = 1+w+w^2$, we see that

$$Q(x,y,z) = 1 + (x+y-z) + (x+y-z)^2$$

8. (18 pts) Let $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$, $P=(0,0)$ and $u^\theta = (\cos\theta, \sin\theta)$, $\theta \in [0,2\pi]$.

(a) Is f continuous at $(0,0)$? Explain.

(b) For fixed θ, write down the definition of the directional derivative $\left(\dfrac{df}{ds}\right)_{u^\theta, P}$ and evaluate it.

(c) Does f have a linear approximation at $(0,0)$? Explain.

Ans:

(a) Let

$$x = r\cos\theta, y = r\sin\theta$$

it follows that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2} = 0 = f(0,0),$$

therefore f is continuous at $(0,0)$.

(b)

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u}^\theta, P} = \lim_{h\to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0,0))}{h}$$

$$= \lim_{h\to 0} \frac{h(\cos^3\theta - \sin^3\theta)}{h} = \cos^3\theta - \sin^3\theta$$

(c) Since $\nabla f(0,0)\cdot \boldsymbol{u}^\theta = (1,1)\cdot(\cos\theta,\sin\theta) \neq \cos^3\theta - \sin^3\theta = \left(\dfrac{df}{ds}\right)_{\boldsymbol{u}^\theta, P}$, it follow from Section 14.5, Theorem 9 that f is not differentiable at $(0,0)$. Therefore f does not have a linear approximation.

Method II:

By definition of differentiability, we need to check whether

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-0)^2 + (y-0)^2}} = 0\,? \tag{1}$$

where

$$L(x,y) = f(0,0) + \partial_x f(0,0)\cdot(x-0) + \partial_y f(0,0)\cdot(y-0) \tag{2}$$

since $L(x,y)$ is the only candidate of the linear approximation. From step (b), we have $\partial_x f(0,0) = 1$, $\partial_y f(0,0) = -1$, $L(x,y) = x - y$, it is easy to see from the Two Path Test that the limit in (1) does not exist. Therefore f does not have a linear approximation at $(0,0)$.

9. (12 pts) Use any method to find $T_{\tan^{-1},0}(x)$. Show detailed derivation.

Ans:

$$\tan^{-1}x = \int_0^x \frac{1}{1+t^2}dt$$

$$= \int_0^x \left(1 - t^2 + t^4 \cdots + (-1)^k t^{2k} + \cdots\right) dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^k \frac{x^{2k+1}}{2k+1} + \cdots, \quad |x| < 1$$

Since $\tan^{-1} x$ has a power series representation on a non-empty interval centered at 0, it follows that this power series is the Taylor series of $\tan^{-1} x$ there.

10. (12 pts) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n+1} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n+1}$ on $x = 0$ and $0 < |x| < 1$ using computational rules of power series.

    Answer. The value is 1 as $x = 0$. For $x \neq 0$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{x^n}{n+2} &= x^{-1} \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right) \\
&= x^{-1} \left(\sum_{n=0}^{\infty} \int_0^x t^n \, dt \right) \\
&= x^{-1} \left(\int_0^x \sum_{n=0}^{\infty} t^n \, dt \right) \\
&= x^{-1} \left(\int_0^x \frac{1}{1-t} \, dt \right) \quad \textbf{(2 pts)} \\
&= x^{-1} \left(-\ln(1-x) \right) \\
&= -\frac{\ln(1-x)}{x}.
\end{aligned}$$