Brief solutions to Midterm 1 (**Average = 55.64 pts**)

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1. (8 pts) (**Average = 5.0 pts**)

   Evaluate $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

   Ans: $\frac{1}{6}$. See page 2 of Lecture 11.

2. (12 pts) (**Average = 6.49 pts**)

   For what values of x does $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$ converge?

   Ans: $|x| \le 1$.

   From root test, $\rho = |x|$, therefore the series converges on $|x| < 1$.

   On $x = 1$, the series converges by Integral Test.

   On $x = -1$, the series converges since it converges absolutely from the case $x = 1$ (or from Alternating Series Test).

3. (12 pts) (**Average = 6.82 pts**)

   Find $\sum_{n=1}^{\infty} nx^n$ and $\sum_{n=1}^{\infty} n^2x^n$ on $|x| < 1$ using computational rules for power series (multiplication, differentiation, integration, etc.).

   Ans:

$$1 + x + x^2 + \ldots = \frac{1}{1-x} \quad \textbf{(geometric series)}$$

$$\Rightarrow \quad x + 2x^2 + 3x^3 + \ldots = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2} \quad \textbf{(Term by Term Differentiation)}$$

$$\Rightarrow \quad x + 2^2x^2 + 3^2x^3 + \ldots = x\left(\frac{x}{(1-x)^2}\right)' = \frac{x(1+x)}{(1-x)^3}$$

4. (12 pts) (**Average = 5.19 pts**)

   Give an approximation of $\int_0^{\frac{1}{2}} \cos(x^2)dx$ to within 10^{-5}. Give the formula of the approximation, but need not find the numerical value. Explain why the error is less than 10^{-5}.

   Ans:

$$\cos x^2 = 1 - \frac{1}{2!}(x^2)^2 + \frac{1}{4!}(x^2)^4 - \frac{1}{6!}(x^2)^6 + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{(x^2)^{2k}}{(2k)!}$$

From the error estimate for alternating series:

$$\left|\cos x^2 - \sum_{k=0}^{n}(-1)^k\frac{(x^2)^{2k}}{(2k)!}\right| \le \left|\frac{(x^2)^{2n+2}}{(2n+2)!}\right|$$

$$\text{error} = \left|\int_0^{\frac{1}{2}} \cos x^2\,dx - \sum_{k=0}^{n}\int_0^{\frac{1}{2}}(-1)^k\frac{(x^2)^{2k}}{(2k)!}\,dx\right| \le \int_0^{\frac{1}{2}}\frac{(x^2)^{2n+2}}{(2n+2)!}\,dx$$

We want

$$\int_0^{\frac{1}{2}}\frac{(x^2)^{2n+2}}{(2n+2)!}\,dx = \frac{1}{(4n+5)2^{4n+5}(2n+2)!} < 10^{-5}$$

Any $n \ge 1$ will do, since for $n=1$, $(4n+5)2^{4n+5}(2n+2)! = 9\cdot 2^9\cdot 4! > 10^6$

Therefore the approximation is

$$\sum_{k=0}^{1}\int_0^{\frac{1}{2}}(-1)^k\frac{(x^2)^{2k}}{(2k)!}\,dx = \sum_{k=0}^{1}(-1)^k\frac{1}{(4k+1)\cdot 2^{4k+1}\cdot(2k)!} = \frac{1}{2} - \frac{1}{320}$$

5. (6+12 pts) (**Average = 5.93 + 4.53 pts**)

(a) Show that the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\cdot n!}$ converges *absolutely*.

(b) Find the sum of the series in (a). Prove your answer (i.e., why the equality holds).

Ans:

(a) Ratio test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{2^n n!}} = 0 < 1, \quad \text{the series converges absolutely. } \textbf{(6 pts)}$$

(b) Sum $= e^{-1/2}$. **(4 pts)** Since

$$1 - \frac{1}{2\cdot 1!} + \frac{1}{4\cdot 2!} - \cdots + (-1)^n\frac{1}{2^n\cdot n!} + \cdots = T_{e^x,0}(x = -\frac{1}{2})$$

$$T_{e^x,0}(-\frac{1}{2}) = e^{-\frac{1}{2}} \text{ if and only if } \lim_{n\to\infty} R_n(-\frac{1}{2}) = 0,$$

$$|R_n(-\frac{1}{2})| \le \frac{e^{c_{n+1}}}{(n+1)!}\frac{1}{2^{n+1}} \le \frac{1}{(n+1)!}\frac{1}{2^{n+1}} \text{ since } c_{n+1}\in(-\frac{1}{2},0)\ .$$

It follows that $\lim_{n\to\infty} R_n(-\frac{1}{2}) = 0$. Therefore $T_{e^x,0}(-\frac{1}{2}) = e^{-\frac{1}{2}}$.**(8 pts)**

6. (6+6+6 pts, 6+9+9 pts with extra points) (**Average = 3.80 + 7.23 + 2.52 pts**)

True or False? Prove it if true, give a counter example if false.

(a) If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on $|x-a| < R$, $R > 0$, then $T_{f,a}(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(c) If $g(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ on $|x| < 1$, then $f(x) = g(x)$ on $|x| < 1$.

Ans: (**3 pts** for correct True or correct False. **3 pts** for correct explanation.)

(a) True. From Term by Term Differentiation Theorem, we get

$$f^{(n)}(a) = n!c_n.$$

Therefore

$$T_{f,a}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} a_n(x-a)^n$$

(b) True. Denote by $L = \sum_{n=1}^{\infty} a_n$ and $S_n = \sum_{k=1}^{n} a_k$. Since $a_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0.$$

(c) False. Take $f(x) = e^{-1/x^2}$ for $x \neq 0$, and $f(x) = 0$ for $x = 0$. Then $f^{(n)}(0) = 0$, $\forall\, n \geq 0$. Therefore $f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 0 \neq f(x)$, if $x \neq 0$.

7. (8 pts) (**Average = 4.40 pts**)

Use any method to find $T_{\tan,0}(x)$ upto x^5 term.

Ans: Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

Since $\cos 0 \neq 0$, $\tan x = \dfrac{\sin x}{\cos x}$ can also be represented by a power series on $|x - 0| < \delta$. This series can be computed using undetermined coefficient or long division to get

$$\tan x = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}{1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 - \cdots} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots,$$

This power series also equals $T_{\tan,0}(x)$ from problem 6(a).

Direct evaluation using derivatives of $\tan x$ at $x = 0$ also gets full credit as long as the results are correct.

8. (12 pts) (**Average = 5.06 pts**)

Use any method to find $T_{\sin^{-1},0}(x)$. Then find the radius of convergence of this series.

Hint: Binomial series.

Ans:

$$\sin^{-1} x = \int_0^x (1-t^2)^{\frac{-1}{2}}\,dt$$

$$= \int_0^x \left(1 - \frac{1}{2}(-t^2) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(-t^2)^2 + \cdots + (-1)^n \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n!}(-t^2)^n + \cdots\right) dt$$

$$= x + \frac{\frac{1}{2}}{1!}\frac{x^3}{3} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\frac{x^5}{5} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\frac{x^7}{7} + \cdots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n!}\frac{x^{2n+1}}{2n+1} + \cdots$$

Radius of convergence = 1 by Ratio Test.