Brief solutions to selected problems in homework 13

1. Section 16.3: Solutions, common mistakes and corrections:

Sec 16.3

21. $\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\,dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy + \left(-\frac{y}{z^2}\right)dz.$

Find f satisfys $\nabla f = \left(\frac{1}{y}, \frac{1}{z} - \frac{x}{y^2}, -\frac{y}{z^2}\right)$

$\Rightarrow f_x = \frac{1}{y} \Rightarrow f = \frac{x}{y} + g(y,z)$ —① for some $g: \mathbb{R}^2 \to \mathbb{R}$.

$f_y = \frac{1}{z} - \frac{x}{y^2}$, by ① $f_y = -\frac{x}{y^2} + g_y \Rightarrow g_y = \frac{1}{z} \Rightarrow g = \frac{y}{z} + h(z)$ —②

$f_z = -\frac{y}{z^2}$, by ①② $f_z = 0 + \left(-\frac{y}{z^2}\right) + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = C.$

for some constant C.

$\Rightarrow f = \frac{x}{y} + \frac{y}{z} + C.$ take $f = \frac{x}{y} + \frac{y}{z}$.

$\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\,dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy + \left(-\frac{y}{z^2}\right)dz = \left.\frac{x}{y} + \frac{y}{z}\right|_{(1,1,1)}^{(2,2,2)} = 0.$

Figure 1: Section 16.3, problem 21

26. find f s.t. $\nabla f = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

take $f = \sqrt{x^2+y^2+z^2}$.

Figure 2: Section 16.3, problem 26

2. Problem 2:

$$F = \frac{x}{\sqrt{x^2+y^2}}\,i + \frac{y}{\sqrt{x^2+y^2}}\,j + 0\,k$$

$$G = \frac{-y}{x^2+y^2}\,i + \frac{x}{x^2+y^2}\,j + 0\,k$$

a $M_1 = \frac{x}{\sqrt{x^2+y^2}}$, $N_1 = \frac{y}{\sqrt{x^2+y^2}}$, $P_1 = 0$

$$\frac{\partial P_1}{\partial y} = 0 = \frac{\partial N_1}{\partial z},\ \frac{\partial P_1}{\partial x} = 0 = \frac{\partial M_1}{\partial z},\ \frac{\partial M_1}{\partial y} = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} = \frac{\partial N_1}{\partial x}$$

$\Rightarrow F$ satisfies the component test

$M_2 = \frac{-y}{x^2+y^2}$, $N_2 = \frac{x}{x^2+y^2}$, $P_2 = 0$

$$\frac{\partial P_2}{\partial y} = 0 = \frac{\partial N_2}{\partial z},\ \frac{\partial P_2}{\partial x} = 0 = \frac{\partial M_2}{\partial z},\ \frac{\partial M_2}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial N_2}{\partial x}$$

$\Rightarrow G$ satisfies the component test

Figure 3: homework 13, problem 2(a)

b $\nabla f = F$

$$\frac{\partial f}{\partial x} = M_1,\ \frac{\partial f}{\partial y} = N_1,\ \frac{\partial f}{\partial z} = P_1$$

$$f(x,y,z) = \sqrt{x^2+y^2} + g(y,z)$$

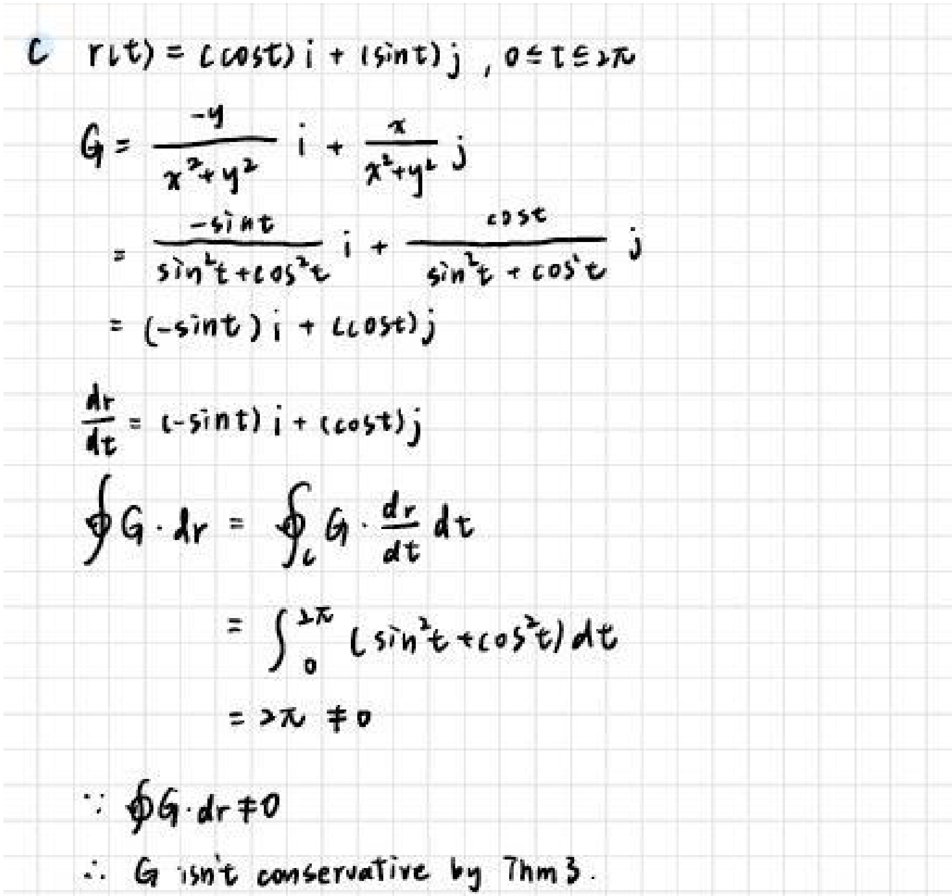
$$\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial g}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow \frac{\partial g}{\partial y} = 0$$

$$\Rightarrow f(x,y,z) = \sqrt{x^2+y^2} + h(z)$$

$$0 + \frac{\partial h}{\partial z} = 0 \Rightarrow \frac{\partial h}{\partial z} = 0,\ h(z) = \cancel{z+C}\ C$$

$$\Rightarrow f(x,y,z) = \sqrt{x^2+y^2} \cancel{+z} + C$$

Figure 4: homework 13, problem 2(b)



C $r(t) = (\cos t)\,i + (\sin t)\,j$, $0 \le t \le 2\pi$

$$G = \frac{-y}{x^2+y^2}\,i + \frac{x}{x^2+y^2}\,j$$

$$= \frac{-\sin t}{\sin^2 t+\cos^2 t}\,i + \frac{\cos t}{\sin^2 t+\cos^2 t}\,j$$

$$= (-\sin t)\,i + (\cos t)\,j$$

$$\frac{dr}{dt} = (-\sin t)\,i + (\cos t)\,j$$

$$\oint G \cdot dr = \oint_C G \cdot \frac{dr}{dt}\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= 2\pi \ne 0$$

$\because \oint G \cdot dr \ne 0$

$\therefore$ G isn't conservative by Thm 3.

Figure 5: homework 13, problem 2(c)

Problem 2(d): It is easier to explain the idea if we restrict problem 2 in the plane:

Let $\boldsymbol{F} = \dfrac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{j}$ and $\boldsymbol{G} = \dfrac{-y}{x^2+y^2}\,\boldsymbol{i} + \dfrac{x}{x^2+y^2}\,\boldsymbol{j}$.

(a) Show that both $\boldsymbol{F}$ and $\boldsymbol{G}$ satisfy the component test.

(b) The natural domain of both $\boldsymbol{F}$ and $\boldsymbol{G}$ is $\{(x,y),\, x^2+y^2 \ne 0\}$ (that is where $\boldsymbol{F}$ and $\boldsymbol{G}$ are defined). Show that $\boldsymbol{F}$ is conservative in this domain by finding its potential function.

(c) Show that $\boldsymbol{G}$ is NOT conservative in this domain (see Example 5 on p990).

(d) If given another $\boldsymbol{H}$ satisfying the component test in this domain, how do you determine whether $\boldsymbol{H}$ is conservative?

Ans: It is clear that answers to (a), (b), (c) remain unchanged.

For (d): Suppose $\boldsymbol{H}$ satisfies the component test in $\{(x,y),\, x^2+y^2 \ne 0\}$. Let C be any simple closed curve, and $\mathcal{R}$ be the inside of C.

(a) If $(0,0) \notin \mathcal{R}$.

In this case, $\mathcal{R}$ is simply connected. We can apply the 2D version of 'Component Test for Conservative Field" statement on page 988, to conclude that ($\boldsymbol{H}$ is conservative, and therefore)

$$\oint_C \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0 \tag{1}$$

(b) If $(0,0) \in \mathcal{R}$.
As explained in Lecture 28, page 12, we have

$$\oint_C \boldsymbol{H} \cdot \boldsymbol{T}\, ds = \oint_{C_a} \boldsymbol{H} \cdot \boldsymbol{T}\, ds \tag{2}$$

where $C_a = \{(x,y),\ x^2 + y^2 = a^2\}$. Moreover, it is clear that the line integral in (2) is independent of $a > 0$.

We conclude from the above analysis that,

(a) If $\displaystyle\oint_{C_a} \boldsymbol{H} \cdot \boldsymbol{T}\, ds \neq 0$, then from Theorem 3 (loop property), $\boldsymbol{H}$ is not conservative.

(b) If $\displaystyle\oint_{C_a} \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0$, then we conclude from (1), (2) that

$$\oint_C \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0 \tag{3}$$

for every simple closed curve C.
If C is closed but not simple (i.e. C intersects itself), we can always decompose C into several simple closed curves (break up at the intersection points and reconnect), it follows that (3) remains valid even if C is not simple closed.

In summary, we have the following conclusion:

$$\boldsymbol{H} \text{ is conservative} \iff \oint_C \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0 \text{ for any closed curve } C \iff \oint_{C_a} \boldsymbol{H} \cdot \boldsymbol{T}\, ds = 0 \tag{4}$$

The conclusion (4) remains valid in 3D. The argument is similar, with the following replacement of key words:

2D: If C is simple closed and $(0,0) \notin \Omega$. (3D: If C does not circle around the z-axis).

2D: If C is simple closed and $(0,0) \in \Omega$. (3D: If C circles around the z-axis once).

2D: $C_a = \{(x,y),\ x^2 + y^2 = a^2\}$. (3D: $C_a = \{(x,y,z=0),\ x^2 + y^2 = a^2\}$).

2D: If C is not simple closed. (3D: If C circles around the z-axis more than once).

3. Problem 3:

2. Let $\vec{F} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x, y, z)$.

(a) What is the natural domain D_F of $\vec{F}$?

(b) Show that $\vec{F}$ satisfies component test in D_F.

(c) Is D_F simply connected?

(d) Is $\vec{F}$ conservative in this domain?

(a) $D_F = \{(x, y, z) \mid x^2+y^2+z^2 > 0\} = \mathbb{R}^3 \setminus \{(0,0,0)\}$

(b)

$$\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = -xy(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = -xz(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = -yz(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$\therefore$ $\vec{F}$ satisfies component test in D_F

(c)

D_F is simply connected

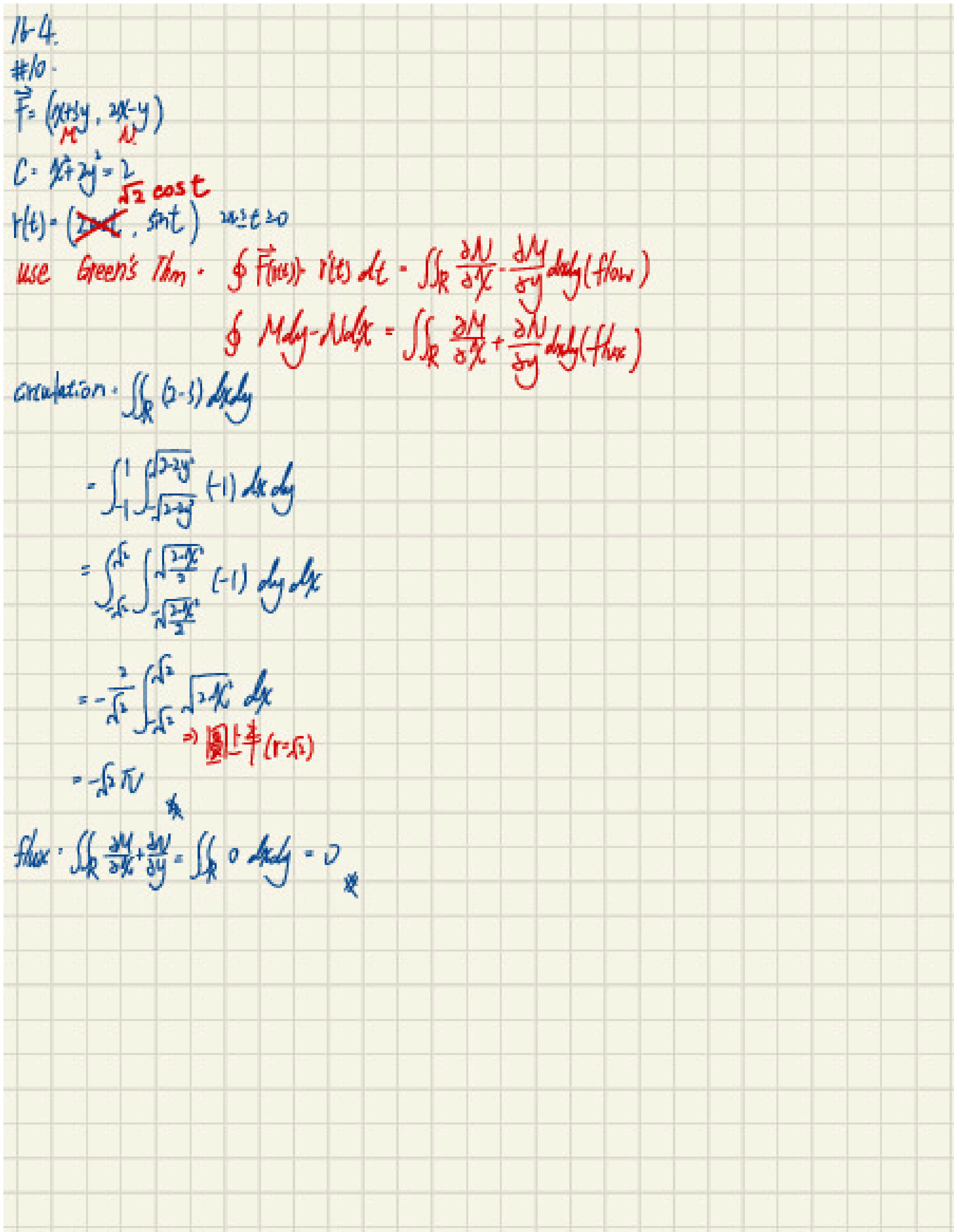
(d) By (b), $\vec{F}$ satisfies component test in D_F.

Also, D_F is simply connected.

$\therefore$ $\vec{F}$ is conservative in D_F

Method 2: By observation (or whatever methods), we know that $\boldsymbol{F} = \nabla\sqrt{x^2+y^2+z^2}$, therefore $\boldsymbol{F}$ is conservative.

4. Section 16.4: Solutions, common mistakes and corrections:



16-4.
#10.
$\vec{F} = (\underbrace{x+3y}_{M}, \underbrace{2x-y}_{N})$
$C: x^2+2y^2=2$
$r(t) = (\cancel{2\cos t}\ \sqrt{2}\cos t, \sin t)$ $2\pi \ge t \ge 0$
use Green's Thm: $\oint \vec{F}(r(t))\cdot r'(t)\,dt = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dxdy$ (flow)
$\oint M\,dy - N\,dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dxdy$ (flux)
circulation: $\iint_R (2-3)\,dxdy$
$= \int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} (-1)\,dx\,dy$
$= \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{\frac{2-x^2}{2}}}^{\sqrt{\frac{2-x^2}{2}}} (-1)\,dy\,dx$
$= -\frac{2}{\sqrt{2}}\int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{2-x^2}\,dx$
⇒ 圓上半 $(r=\sqrt{2})$
$= -\sqrt{2}\pi$ ※
flux: $\iint_R \frac{\partial M}{\partial x}+\frac{\partial N}{\partial y} = \iint_R 0\,dxdy = 0$ ※

Figure 6: Section 16.4, problem 10

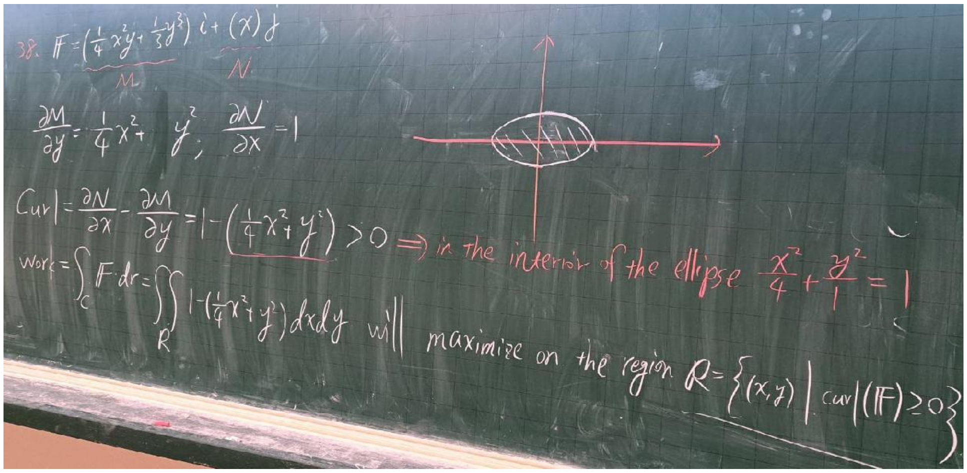
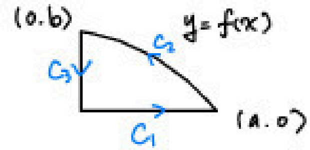


Figure 7: Section 16.4, problem 38

5. Problem 5:

4. $F = (M(x,y), \underbrace{0}_{N=0})$ on R

Tangential form :



$C_1 : \begin{cases} x = t \\ y = 0 \end{cases} \quad 0 \le t \le a$

$C_2 : \begin{cases} x = a-t \\ y = f(a-t) \end{cases} \quad 0 \le t \le a$

$C_3 : \begin{cases} \boxed{x=0} \\ y = b-t \end{cases} \quad 0 \le t \le b$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$= \int_0^a \int_0^{f(x)} -\frac{\partial M}{\partial y}\, dy\,dx$$

$$= \int_0^a \left(-M(x.f(x)) + M(x.0)\right) dx$$

$$= -\int_0^a M(x.f(x))\,dx + \int_0^a M(x.0)\,dx$$

$$\oint_C M dx - N dy$$

$$= \oint_{C_1} M\,dx + \oint_{C_2} M dx + \oint_{C_3} \cancel{M dx} \rightarrow 0$$

$$= \int_0^a M(t,0)\,dt + \int_0^a M(a-t, f(a-t))(-dt)$$

let $x = a-t$
$dx = -dt$

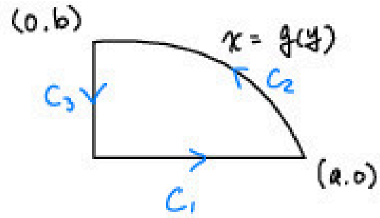
$$= \int_0^a M(t,0)dt + \int_a^0 M(x.f(x))\,dx$$

$$= \int_0^a M(x.0)\,dx - \int_0^a M(x.f(x))\,dx$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy = \oint_C M dx + N dy$$

Figure 8: homework 13, problem 5, tangential form

Normal form:



$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \int_0^b \int_0^{g(y)} \frac{\partial M}{\partial x}\, dx\, dy$$

$$= \int_0^b \big(M(g(y), y) - M(0, y)\big)\, dy$$

$$= \int_0^b M(g(y), y)\, dy - \int_0^b M(0, y)\, dy$$

$C_1: \begin{cases} x = t \\ \boxed{y = 0} \end{cases} \quad 0 \le t \le a$

$C_2: \begin{cases} x = g(t) \\ y = t \end{cases} \quad 0 \le t \le b$

$C_3: \begin{cases} x = 0 \\ y = b - t \end{cases} \quad 0 \le t \le b$

$$\oint_C M\,dy - N\,dx$$

$$= \oint_{C_1} \underbrace{M\,dy}_{0} + \oint_{C_2} M\,dy + \oint_{C_3} M\,dy$$

$$= \int_0^b M(g(t), t)\, dt + \int_0^b M(0, b-t)(-dt)$$

let $y = b - t$
$dy = -dt$

$$= \int_0^b M(g(t), t)\, dt + \int_b^0 M(0, y)\, dy$$

$$= \int_0^b M(g(y), y)\, dt - \int_0^b M(0, y)\, dy$$

$$\therefore \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \oint_C M\,dy - N\,dx$$

Figure 9: homework 13, problem 5, normal form