Brief solutions to selected problems in homework 02 (part 2)

1. Section 10.4, problem 61: $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^q}{n^p}$, $p>1$.

Answer:

case I: $p>1$, $q\le 0$.

Take $p=1.5$, $q=-2.3$ for example. Since $\ln n>1$ for $n\ge 3$, we have

$$\sum_{n=3}^{\infty}\frac{(\ln n)^q}{n^p}=\sum_{n=3}^{\infty}\frac{1}{(\ln n)^{2.3}n^{1.5}}<\sum_{n=3}^{\infty}\frac{1}{n^{1.5}}<\infty.$$

Therefore $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^{-2.3}}{n^{1.5}}$ converges by the Comparison Test.

The same argument works for any $p>1$, $q\le 0$. Just replace 1.5 by p and -2.3 by q.

case II: $p>1$, $q>0$.

Take $p=3.2$, $q=4.6$ for example. Let $a_n=\dfrac{(\ln n)^{3.2}}{n^{4.6}}$.

Since $a_n>\frac{1}{n^{4.6}}$ for $n\ge 3$, comparing $\displaystyle\sum_{n=2}^{\infty}a_n$ with $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^{4.6}}$ (convergent) leads to no conclusion.

We need to compare a_n with $b_n=\dfrac{1}{n^r}$ by choosing an r so that $1<r<p$. Therefore we take $r=\dfrac{1+p}{2}=2.1$, $b_n=\dfrac{1}{n^{2.1}}$, and apply the Limit Comparison Test:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{(\ln n)^q}{n^{p-r}}=\lim_{n\to\infty}\frac{(\ln n)^{4.6}}{n^{3.2-2.1}}$$

Instead of applying L'Hôpital's Rule to $\displaystyle\lim_{n\to\infty}\frac{(\ln n)^q}{n^{p-r}}=\lim_{n\to\infty}\frac{(\ln n)^{4.6}}{n^{3.2-2.1}}$ directly, we notice that

$$\lim_{n\to\infty}\frac{(\ln n)^q}{n^{p-r}}=\lim_{n\to\infty}\left(\frac{\ln n}{n^{\frac{p-r}{q}}}\right)^q=\left(\lim_{n\to\infty}\frac{\ln n}{n^{\frac{p-r}{q}}}\right)^q=\left(\lim_{n\to\infty}\frac{\ln n}{n^{\frac{3.2-2.1}{4.6}}}\right)^{4.6}$$

The limit $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n^{\frac{p-r}{q}}}$ is easier to compute. By L'Hôpital's Rule:

$$\lim_{n\to\infty}\frac{\ln n}{n^{\frac{p-r}{q}}}=\lim_{n\to\infty}\frac{\frac{1}{n}}{(\frac{p-r}{q})n^{\frac{p-r}{q}-1}}=\lim_{n\to\infty}\frac{1}{(\frac{p-r}{q})n^{\frac{p-r}{q}}}=\lim_{n\to\infty}\frac{1}{(\frac{3.2-2.1}{4.6})n^{\frac{3.2-2.1}{4.6}}}=0$$

Therefore

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\left(\lim_{n\to\infty}\frac{\ln n}{n^{\frac{p-r}{q}}}\right)^q=0$$

Since $\displaystyle\sum_{n=2}^{\infty}b_n$ converges, we know from the Comparison Test that $\displaystyle\sum_{n=2}^{\infty}a_n$ also converges. Again, the same argument works for any $p>1$, $q>0$ and $1<r<p$.

2. Section 10.4, problem 62: $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^q}{n^p}$, $0 < p < 1$.

   Answer: The proof for problem 62 is similar:

   case III: $0 < p < 1$, $q \geq 0$.

   Compare it with $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^p}$:

$$\sum_{n=3}^{\infty}\frac{(\ln n)^q}{n^p} > \sum_{n=3}^{\infty}\frac{1}{n^p}$$

   Since $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^p} = \infty$ for $0 < p < 1$, we know by the Comparison Test that $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^q}{n^p}$ diverges.

   case IV: $0 < p < 1$, $q < 0$.

   Compare it with $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^r}$, $p < r < 1$ (take $r = \frac{p+1}{2}$ for example). The rest of the calculation is similar to case II and leads to the conclusion that $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^q}{n^p}$ diverges.