THEOREM 9—The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

EXAMPLE 3 Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

Take $f(x) = \frac{x}{x^2+4}$ which is cont., positive. on $(2,\infty)$

Check: f is decreasing on some interval.

$$f'(x) = \frac{x^2+4-x(2x)}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2} < 0 \text{ when } x>2$$

$\Rightarrow f$ is decreasing on $(2,\infty)$

Note that $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ and $\sum_{n=3}^{\infty} \frac{n}{n^2+4}$ are both conv. or div.

Compute $\int_3^{\infty} \frac{x}{x^2+4}dx$: $\int_3^{a} \frac{x}{x^2+4}dx = \left(\frac{1}{2}\ln|x^2+4|\right)\Big|_3^a = \frac{1}{2}\left(\ln|a^2+4| - \ln 13\right) = \frac{1}{2}\ln\frac{a^2+4}{13}$

$$\Rightarrow \int_3^{\infty} \frac{x}{x^2+4}dx = \lim_{a\to\infty}\int_3^{a} \frac{x}{x^2+4}dx = \frac{1}{2}\lim_{a\to\infty}\ln\left(\frac{a^2+4}{13}\right) \text{ diverges}$$

By Integral Test, $\sum_{n=3}^{\infty} \frac{n}{n^2+4}$ div. $\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^2+4}$ div.

27. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$

$\because \lim_{n\to\infty} \sqrt{n}$ and $\lim_{n\to\infty} \ln n$ are both ∞

$\therefore$ Apply L'Hôpital's rule to $\lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\ln n} \Rightarrow \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\ln n} \xlongequal[\text{Rule}]{\text{L'Hôpital's}} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$

By n-th Term Test, $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ diverges.

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

Note that $\dfrac{1}{\sqrt{n}(\sqrt{n}+1)} = \dfrac{1}{\sqrt{n}} - \dfrac{1}{\sqrt{n}+1}$

Take $f(x) = \dfrac{1}{\sqrt{x}(\sqrt{x}+1)} = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{x}+1}$ which is positive, cont. on $[1,\infty)$

Check: f decreases.

$$f'(x) = -\frac{\frac{1}{2\sqrt{x}}+1}{(x+\sqrt{x})^2} < 0, \quad \forall x > 0$$

$\Rightarrow f$ decreases on $[1,\infty)$

Compute: $\displaystyle\int_1^{\infty} f(x)\,dx$

$\displaystyle\int x^{-\frac{1}{2}}\,dx = 2\sqrt{x}$

$2\sqrt{x} = 2(u-1)$

Let $u = 1+\sqrt{x}$ $\quad \displaystyle 2\int \frac{u-1}{u}\,du = 2\int 1-\frac{1}{u}\,du = 2(u-\ln|u|)$

$\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}} \quad dx = 2\sqrt{x}\,du \qquad = 2(1+\sqrt{x}-\ln|1+\sqrt{x}|)$

$$\int_1^a f(x)\,dx = \int_1^a \left(x^{-\frac{1}{2}} - \frac{1}{1+\sqrt{x}}\right)dx = 2\sqrt{x}\Big|_1^a - \int_1^a \frac{1}{1+\sqrt{x}}\,dx = 2\sqrt{a}-2-2\Big[\big(1+\sqrt{a}-\ln(1+\sqrt{a})\big)-(2-\ln 2)\Big]$$

$$= \cancel{2\sqrt{a}}\cancel{-2-2}\cancel{-2\sqrt{a}}+2\ln(1+\sqrt{a})\cancel{+4}-2\ln 2 = 2\ln(1+\sqrt{a})-2\ln 2 \to \infty$$

as $a\to\infty$

Hence $\displaystyle\int_1^{\infty} f(x)\,dx$ div.

By Integral Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ div.

Method II: Let $a_n = \dfrac{1}{\sqrt{n}(\sqrt{n}+1)} > 0$ and $b_n = \dfrac{1}{n} > 0$, $\forall n \in \mathbb{N}$.

$\because \displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = 1 > 0$. and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ div. by p-series.

$\therefore$ By Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ div.

31. $\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$

Take $f(x)=\frac{1}{x\ln x\sqrt{(\ln x)^2-1}}$. (cont., positive, decreasing)

$$\int_3^{\infty} \frac{\frac{1}{x}}{\ln x\sqrt{(\ln x)^2-1}}\,dx = \lim_{a\to\infty}\int_{\ln 3}^{a}\frac{1}{u\sqrt{u^2-1}}\,du = \lim_{a\to\infty}\left(\sec^{-1}|u|\right)_{\ln 3}^{a} = \lim_{a\to\infty}\left(\cos^{-1}\frac{1}{a}-\sec^{-1}(\ln 3)\right)$$

$$= \cos^{-1}0-\sec^{-1}\ln 3 \text{ conv.}$$

Let $u=\ln x$ $\Rightarrow du=\frac{1}{x}dx$

$$\Rightarrow \sum_{n=3}^{\infty}\frac{\frac{1}{n}}{\ln n\sqrt{(\ln n)^2-1}} \text{ conv.}$$

33. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

Since $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{m\to 0} \frac{\sin m}{m} = 1 \neq 0$,

By n-th Term Test, $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$ div.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$

Take $f(x) = \dfrac{8\tan^{-1}x}{1+x^2}$ $\Rightarrow$ f is positive, cont. on $[1,\infty)$

Check: f decreases on $[1,\infty)$

$$f'(x) = \frac{\frac{8}{\cancel{1+x^2}}\cancel{(1+x^2)} - 8\tan^{-1}x \cdot 2x}{(1+x^2)^2} = \frac{8(1-\tan^{-1}x \cdot 2x)}{(1+x^2)^2} < 0,\ \forall x \in [1,\infty)$$

Compute: $\displaystyle\int_1^{\infty} \frac{8\tan^{-1}x}{1+x^2}\,dx$

$$\int_1^a \frac{8\tan^{-1}x}{1+x^2}\,dx = 8\int_1^a \frac{\tan^{-1}x}{1+x^2}\,dx = \overset{4}{\cancel{8}}\cdot\left(\frac{1}{2}(\tan^{-1}x)^2\right)\Big|_1^a = 4\left((\tan^{-1}a)^2 - \tan^{-1}1\right)$$ converges when $a \to \infty$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1}n}{1+n^2}$ conv.

For what values of a, if any, do the series in Exercises 41 and 42 converge?

41. $\sum_{n=1}^{\infty}\left(\frac{a}{n+2}-\frac{1}{n+4}\right)$ 42. $\sum_{n=3}^{\infty}\left(\frac{1}{n-1}-\frac{2a}{n+1}\right)$

Let $f(x)=\frac{a}{x+2}-\frac{1}{x+4}$ (Note that f is cont. $\forall x>0$)

① $\frac{a}{x+2}-\frac{1}{x+4}>0 \Rightarrow a>\frac{x+2}{x+4}\longrightarrow 1$ as $x\to\infty$. $\Rightarrow a\geq 1$

② $f'(x)=-\frac{a}{(x+2)^2}+\frac{1}{(x+4)^2}<0 \Rightarrow a>\frac{(x+2)^2}{(x+4)^2}\to 1$ as $x\to\infty \Rightarrow a\geq 1$

Using Integral Test, first compute $\int_1^k \frac{a}{x+2}-\frac{1}{x+4}\,dx = \ln\frac{(k+2)^a}{k+4}-\ln\frac{3^a}{5}$

Want $\ln\frac{(k+2)^a}{k+4}$ has limit as $k\to\infty$, we must have $a\leq 1$ ($\because$ 要麻分子.分母 power 一樣, 要麻分母 power > 分子)

$\Rightarrow a=1$

Method II: Note that $\frac{a}{n+2}-\frac{1}{n+4}=\frac{(a-1)n+(4a-2)}{(n+2)(n+4)}$. We can see that if $a=1$, $\sum_{n=1}^{\infty}\left(\frac{a}{n+2}-\frac{1}{n+4}\right)$ conv.

Use Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n+2}$ which is div.

We get: If $a\neq 1$, $\sum_{n=1}^{\infty}\left(\frac{a}{n+2}-\frac{1}{n+4}\right)$ div

45. Is it true that if $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then there is also a divergent series $\sum_{n=1}^{\infty} b_n$ of positive numbers with $b_n < a_n$ for every n? Is there a "smallest" divergent series of positive numbers? Give reasons for your answers.

Yes, if $\sum_{n=1}^{\infty} a_n$ div. with $a_n > 0$, $\forall n \in \mathbb{N}$, then $b_n := \frac{a_n}{2} < a_n$ and $\sum_{n=1}^{\infty} b_n = \frac{1}{2}\sum_{n=1}^{\infty} a_n$ also div.

There is no "smallest" since you can divide a_n by any real positive number.

55. Logarithmic *p*-series

a. Show that the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} \quad (p \text{ a positive constant})$$

converges if and only if $p > 1$.

b. What implications does the fact in part (a) have for the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}?$$

Give reasons for your answer.

a. Compute $\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx \xlongequal[du=\frac{1}{x}dx]{\text{Let } u=\ln x} \int_{\ln 2}^{\infty} u^{-p}\,du = \lim_{a\to\infty}\left(\frac{u^{-p+1}}{-p+1}\right)\Big|_{\ln 2}^{a} = \lim_{a\to\infty}\left(\frac{1}{1-p}\cdot\left(a^{1-p}-(\ln 2)^{1-p}\right)\right) = \begin{cases} \frac{(\ln 2)^{1-p}}{p-1}, & \text{if } p>1. \\ \text{div.}, & \text{if } p<1. \end{cases}$

As for $p=1$: $\int_2^{\infty} \frac{1}{x\ln x}\,dx \xlongequal[du=\frac{1}{x}dx]{\text{Let } u=\ln x} \int_{\ln 2}^{\infty} u^{-1}\,du = \lim_{a\to\infty}\left(\ln(\ln x)\right)\Big|_2^{a}$ div.

Hence, $\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx$ converges iff $p>1$.

b. Since $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ and $\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx$ converge or diverge together, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges iff $p>1$.

THEOREM 10—The Comparison Test Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \le a_n \le c_n \qquad \text{for all} \qquad n > N.$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

THEOREM 11—Limit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer).

1. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

15. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

(*Hint:* Limit Comparison with $\sum_{n=2}^{\infty} (1/n)$)

Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, by p-series, we have $\sum_{n=2}^{\infty} \frac{1}{n}$ div.

And $\frac{1}{\ln n}$, $\frac{1}{n} > 0$, $\forall n \in \mathbb{N}, n \geq 2$, compute $\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} \underset{\text{rule}}{\overset{\text{L'Hôpital's}}{=}} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty \Rightarrow$ By Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ div.

16. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

($Hint$: Limit Comparison with $\sum_{n=1}^{\infty} (1/n^2)$)

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, by p-series, it's conv.

Compute $\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n^2})}{\frac{1}{n^2}} \overset{\text{L'Hopitals}}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}} \cdot \cancel{\left(-\frac{2}{n^3}\right)}}{\cancel{\left(\frac{-2}{n^3}\right)}} = 1$

$\Rightarrow$ By Limit Comparison Test, $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ conv.

17. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, by p-series, it's div.

Compute $\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}+\sqrt[3]{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+\sqrt[3]{n}} = \lim_{n\to\infty} \frac{1}{2+n^{-\frac{1}{6}}} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{\sqrt[6]{n}}} = \frac{1}{2}$

By Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$ div.

27. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

$\because n > \ln n$, $\therefore \ln n > \ln(\ln n)$

$\Rightarrow n > \ln n > \ln(\ln n) \Rightarrow \frac{1}{n} < \frac{1}{\ln n} < \frac{1}{\ln(\ln n)}$

And $\because \sum_{n=3}^{\infty} \frac{1}{n}$ div., $\therefore$ By Direct Comparison Test, we have $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ div.

29. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ which is div..

Compute $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}\ln n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} \xlongequal[\text{rule}]{\text{L'Hopitals}} \lim_{n\to\infty} \frac{\frac{1}{2}\frac{1}{\sqrt{n}}}{\frac{1}{n}} = \frac{1}{2}\lim_{n\to\infty}\sqrt{n}$ div.

$\Rightarrow$ By Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ div.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$

Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which is div.

Compute $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{1+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1+\ln n} \xrightarrow[\text{rule}]{\text{L'Hopital's}} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n$ div.

$\Rightarrow$ By Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ div.

43. $\sum_{n=2}^{\infty} \frac{1}{n!}$

(*Hint:* First show that $(1/n!) \le (1/n(n-1))$ for $n \ge 2$.)

Note that $\frac{1}{n-1} \ge \frac{1}{(n-1)!}$, $\forall n \ge 2$, $\therefore \frac{1}{n(n-1)} \ge \frac{1}{n!}$, $\forall n \ge 2$.

We have shown that $\sum_{n=1}^{\infty} \frac{1}{n(n-1)}$ conv.

Hence by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ conv.

45. $\sum_{n=1}^{\infty} \sin \frac{1}{n}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is div.

Compute $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to 0} \frac{\sin n}{n} = 1$

By Limit Comparison Test, $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ div.

51. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$

Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which is div.

Compute $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \underset{\substack{\uparrow \\ \text{have done}}}{=} 1$

By Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ div.

61. Show that $\sum_{n=2}^{\infty}((\ln n)^q/n^p)$ converges for $-\infty < q < \infty$ and $p > 1$.

(*Hint:* Limit Comparison with $\sum_{n=2}^{\infty} 1/n^r$ for $1 < r < p$.)

Let $-\infty < q < \infty$ and $p > 1$. If $q = 0$, then $\sum_{n=2}^{\infty}\frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty}\frac{1}{n^p}$, which is a convergent p-series. If $q \neq 0$, compare with $\sum_{n=2}^{\infty}\frac{1}{n^r}$ where $1 < r < p$, then $\lim_{n\to\infty}\frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n\to\infty}\frac{(\ln n)^q}{n^{p-r}}$, and $p - r > 0$. If $q < 0 \Rightarrow -q > 0$ and $\lim_{n\to\infty}\frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty}\frac{1}{(\ln n)^{-q}n^{p-r}} = 0$. If $q > 0$, $\lim_{n\to\infty}\frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty}\frac{q(\ln n)^{q-1}\left(\frac{1}{n}\right)}{(p-r)n^{p-r-1}} = \lim_{n\to\infty}\frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}}$. If $q - 1 \le 0 \Rightarrow 1 - q \ge 0$ and $\lim_{n\to\infty}\frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}} = \lim_{n\to\infty}\frac{q}{(p-r)n^{p-r}(\ln n)^{1-q}} = 0$, otherwise, we apply L'Hopital's Rule again. $\lim_{n\to\infty}\frac{q(q-1)(\ln n)^{q-2}\left(\frac{1}{n}\right)}{(p-r)^2 n^{p-r-1}} = \lim_{n\to\infty}\frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}}$. If $q - 2 \le 0 \Rightarrow 2 - q \ge 0$ and

$\lim_{n\to\infty}\frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}} = \lim_{n\to\infty}\frac{q(q-1)}{(p-r)^2 n^{p-r}(\ln n)^{2-q}} = 0$; otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $q - k \le 0 \Rightarrow k - q \ge 0$. Thus, after k applications of L'Hopital's Rule we obtain $\lim_{n\to\infty}\frac{q(q-1)\cdots(q-k+1)(\ln n)^{q-k}}{(p-r)^k n^{p-r}} = \lim_{n\to\infty}\frac{q(q-1)\cdots(q-k+1)}{(p-r)^k n^{p-r}(\ln n)^{k-q}} = 0$. Since the limit is 0 in every case, by Limit Comparison Test, the series $\sum_{n=1}^{\infty}\frac{(\ln n)^q}{n^p}$ converges.

62. (*Continuation of Exercise 61.*) Show that $\sum_{n=2}^{\infty}((\ln n)^q/n^p)$ diverges for $-\infty < q < \infty$ and $0 < p < 1$.

(*Hint:* Limit Comparison with an appropriate p-series.)

. Let $-\infty < q < \infty$ and $p \le 1$. If $q=0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. If $q>0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. Then $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^p} = \lim_{n\to\infty} (\ln n)^q = \infty$. If $q<0 \Rightarrow -q>0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$, where $0<p<r<1$. $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{n^{r-p}}{(\ln n)^{-q}}$ since $r-p>0$. Apply L'Hopital's to obtain $\lim_{n\to\infty} \frac{(r-p)n^{r-p-1}}{(-q)(\ln n)^{-q-1}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(r-p)n^{r-p}}{(-q)(\ln n)^{-q-1}}$. If $-q-1 \le 0 \Rightarrow q+1 \ge 0$ and $\lim_{n\to\infty} \frac{(r-p)n^{r-p}(\ln n)^{q+1}}{(-q)} = \infty$, otherwise, we apply L'Hopital's Rule again to obtain $\lim_{n\to\infty} \frac{(r-p)^2 n^{r-p-1}}{(-q)(-q-1)(\ln n)^{-q-2}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}}$. If $-q-2 \le 0 \Rightarrow q+2 \ge 0$ and $\lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}} = \lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}(\ln n)^{q+2}}{(-q)(-q-1)} = \infty$, otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $-q-k \le 0 \Rightarrow q+k \ge 0$. Thus, after k applications of L'Hopital's Rule we obtain $\lim_{n\to\infty} \frac{(r-p)^k n^{r-p}}{(-q)(-q-1)\cdots(-q-k+1)(\ln n)^{-q-k}}$ $= \lim_{n\to\infty} \frac{(r-p)^k n^{r-p}(\ln n)^{q+k}}{(-q)(-q-1)\cdots(-q-k+1)} = \infty$. Since the limit is ∞ if $q>0$ or if $q<0$ and $p<1$, by Limit comparison test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges. Finally if $q<0$ and $p=1$ then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series. For $n \ge 3$, $\ln n \ge 1 \Rightarrow (\ln n)^q \ge 1 \Rightarrow \frac{(\ln n)^q}{n} \ge \frac{1}{n}$. Thus $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$ diverges by Comparison Test. Thus, if $-\infty < q < \infty$ and $p \le 1$, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges.

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

THEOREM 12—The Absolute Convergence Test If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

THEOREM 13—The Ratio Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

THEOREM 14—The Root Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$

19. $\displaystyle\sum_{n=1}^{\infty} n!(-e)^{-n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

17. $\displaystyle\lim_{n\to\infty}\left|\frac{\frac{(n+1)^{\sqrt{2}}}{\cancel{2^{n+1}}\,2}}{\frac{n^{\sqrt{2}}}{\cancel{2^n}}}\right| = \lim_{n\to\infty}\frac{1}{2}\cdot\left(1+\frac{1}{n}\right)^{\sqrt{2}} = \frac{1}{2} < 1 \Rightarrow$ By Ratio Test, $\displaystyle\sum_{n=1}^{\infty}\frac{n^{\sqrt{2}}}{2^n}$ conv.

19. $\displaystyle\lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}^{\,n+1}\,(-e)^{\cancel{-(n+1)}\,-1}}{\cancel{n!}\,(-e)^{\cancel{-n}}}\right| = \lim_{n\to\infty}\frac{(n+1)}{e}$ div. $\Rightarrow$ By Ratio Test, $\displaystyle\sum_{n=1}^{\infty} n!(-e)^{-n}$ div.

21. $\displaystyle\lim_{n\to\infty}\left|\frac{\frac{(n+1)^{10}}{\cancel{10^{n+1}}\,10}}{\frac{n^{10}}{\cancel{10^n}}}\right| = \lim_{n\to\infty}\frac{1}{10}\cdot\left(1+\frac{1}{n}\right)^{10} = \frac{1}{10} < 1 \Rightarrow$ By Ratio Test, $\displaystyle\sum_{n=1}^{\infty}\frac{n^{10}}{10^n}$ conv.

23. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$

25. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(1-\frac{3}{n}\right)^n$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

23. $\dfrac{2+(-1)^n}{1.25^n} = \left(\dfrac{4}{5}\right)^n \left(2+(-1)^n\right) \le \left(\dfrac{4}{5}\right)^n$ and $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ conv. by geometry series.

By Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$ conv.

25. $\displaystyle\lim_{n\to\infty} (-1)^n\left(1-\frac{3}{n}\right)^n$ div., Hence $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(1-\frac{3}{n}\right)^n$ div.

27. $\because n > \ln n$, $\therefore \dfrac{n}{n^{\cancel{3}\,2}} > \dfrac{\ln n}{n^3}$. And $\because \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by p-series,

Hence by Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ conv.

29. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$ ✓

31. $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{n^e}$ ✓

33. $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{n!}$ ✓

29. $\because \frac{1}{n}-\frac{1}{n^2}=\frac{n-1}{n^2}>\frac{1}{2}\left(\frac{1}{n}\right)$ for $n\geq 3$ and $\sum_{n=3}^{\infty}\frac{1}{2n}$ div. by p-series

$\therefore \sum_{n=3}^{\infty}\frac{1}{n}-\frac{1}{n^2}$ div. by Comparison Test, $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n^2}$ div.

31. $\because \lim_{n\to\infty}\frac{e^n}{n^e}$ div., $\therefore$ by n-th Term Test, $\sum_{n=1}^{\infty}\frac{e^n}{n^e}$ div.

33. $\lim_{n\to\infty}\left|\dfrac{\dfrac{\cancel{(n+2)}(n+3)}{\cancel{(n+1)!}\ n+1}}{\dfrac{(n+1)\cancel{(n+2)}}{\cancel{n!}}}\right|=\lim_{n\to\infty}\frac{n+3}{n^2+2n+1}=0<1$, by Ratio Test, $\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{n!}$ conv.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

39. $\displaystyle\sum_{n=2}^{\infty} \frac{-n}{(\ln n)^n}$

35. $\displaystyle\lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(n+4)!}\ n+4}{\cancel{3!}\,\cancel{(n+1)!}\,\cancel{3^{n+1}}\ (n+1)\ 3}}{\dfrac{\cancel{(n+3)!}}{\cancel{3!}\,\cancel{n!}\,\cancel{3^n}}}\right| = \frac{1}{3}\lim_{n\to\infty}\frac{n+4}{n+1} = \frac{1}{3} < 1$. By Ratio Test, $\displaystyle\sum_{n=1}^{\infty}\frac{(n+3)!}{3!n!\,3^n}$ conv.

37. $\displaystyle\lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(n+1)!}\ n+1}{(2n+3)(2n+2)\cancel{(2n+3)!}}}{\dfrac{\cancel{n!}}{\cancel{(2n+1)!}}}\right| = \lim_{n\to\infty}\frac{n+1}{4n^2+10n+6} = 0 < 1$. By Ratio Test, $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{(2n+1)!}$ conv.

39. $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|\frac{-n}{(\ln n)^n}\right|} = \lim_{n\to\infty}\frac{1}{\ln n}\sqrt[n]{n} = 0$. By Root Test, $\displaystyle\sum_{n=1}^{\infty}\frac{-n}{(\ln n)^n}$ conv.

41. $\displaystyle\sum_{n=1}^{\infty} \frac{n!\ln n}{n(n+2)!}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

41. $\dfrac{\cancel{n!}\ln n}{n\cancel{(n+2)!}} = \dfrac{\ln n}{n(n+1)(n+2)} < \dfrac{\cancel{n}}{\cancel{n}(n+1)(n+2)} = \dfrac{1}{(n+1)(n+2)} < \dfrac{1}{n^2}$

$\because \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by p-series, $\therefore$ By Direct Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n!\ln n}{n(n+2)!}$ conv.

43. $\displaystyle\lim_{n\to\infty} \left| \frac{\dfrac{\overset{n+1}{\cancel{(n+1)!}}\,\overset{n+1}{\cancel{(n+1)!}}}{\cancel{(2n+2)!}\,(2n+1)(2n+2)}}{\dfrac{\cancel{(n!)}\cancel{(n!)}}{\cancel{(2n)!}}} \right| = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$, $\therefore$ By Ratio Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ conv.

61. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{4^n 2^n n!}$

$$\lim_{n\to\infty}\left|\frac{\dfrac{\cancel{1\cdot 3\cdots(2n-1)}(2n+1)}{\overset{4}{\cancel{4^{n+1}}}\,\overset{2}{\cancel{2^{n+1}}}\,\cancel{(n+1)!}\,n+1}}{\dfrac{\cancel{1\cdot 3\cdots(2n-1)}}{\cancel{4^n}\cdot\cancel{2^n}\cdot\cancel{n!}}}\right| = \lim_{n\to\infty}\frac{2n+1}{8n+8} = \frac{1}{4} < 1,$$

$\therefore$ By Ratio Test, $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{4^n\cdot 2^n\cdot n!}$.

65. Let $a_n = \begin{cases} n/2^n, & \text{if } n \text{ is a prime number} \\ 1/2^n, & \text{otherwise.} \end{cases}$

Does $\sum a_n$ converge? Give reasons for your answer.

$a_n \le \frac{n}{2^n}$, $\forall n \in \mathbb{N}$. And $\because \lim_{n\to\infty} \left| \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \right| = \frac{1}{2} < 1$, $\therefore$ By Ratio Test, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ conv.

$\Rightarrow$ By Comparison Test, $\sum_{n=1}^{\infty} a_n$ conv.