Brief solutions to selected problems in homework 01

1. Section 10.1, problem 46:

   Answer:

   Since
   $$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n},$$
   and
   $$\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{2^n},$$
   we conclude from the Sandwich Theorem for sequence that
   $$\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0.$$

2. Section 10.1, problem 59:

   Answer:

   Since
   $$\lim_{n\to\infty} n^{\frac{1}{n}} = 1, \quad \lim_{n\to\infty} \ln n = \infty,$$
   we have
   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{n}}} = \infty \text{ (diverges)}.$$

3. Section 10.1, problem 125:

   proof:

   Here $a_n = \frac{\sin n}{n}$, and the limit $L = 0$. For any $\varepsilon > 0$, take an integer $N > \frac{1}{\varepsilon}$. Then for any $n > N$, we have
   $$|a_n - L| = \frac{|\sin n|}{n} \le \frac{1}{n} < \frac{1}{N} < \varepsilon.$$
   This proves that $\lim_{n\to\infty} a_n = 0$.

4. Section 10.2, problem 43:

   Answer:

   From the identity
   $$a_n = \frac{40n}{(2n-1)^2(2n+1)^2} = 5\Big(\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\Big),$$
   we have
   $$s_k = \sum_{n=1}^{k} a_n = 5\Big(\frac{1}{(2\cdot 1-1)^2} - \frac{1}{(2\cdot 2-1)^2} + \frac{1}{(2\cdot 2-1)^2} - \cdots - \frac{1}{(2\cdot k+1)^2}\Big) = 5\Big(1 - \frac{1}{(2k+1)^2}\Big)$$
   Therefore
   $$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = 5.$$

5. Section 10.2, problem 61:

   Answer:

   Since $\lim_{n\to\infty} a_n = \infty \neq 0$, we conclude from the n-th term test that $\sum_{n=1}^{\infty} a_n$ diverges.

6. Section 10.2, problem 65:

   Answer:

   From the identity

$$a_n = \ln\left(\frac{n}{n+1}\right) = \ln n - \ln(n+1),$$

   we have

$$s_k = \sum_{n=1}^{k} a_n = \ln 1 - \ln 2 + \ln 2 - \ln 3 + \cdots - \ln(k+1).$$

   Therefore

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = -\infty \text{ (diverges).}$$

7. Section 10.2, problem 78:

   Answer:

$$\text{The geometric series } \sum_{n=0}^{\infty} (\ln x)^n \text{ converges } \iff |\ln x| < 1 \iff \frac{1}{e} < x < e.$$

   When $\frac{1}{e} < x < e$, $\sum_{n=0}^{\infty} (\ln x)^n = \dfrac{1}{1-\ln x}$.