Brief solutions to Final Exam (**Average = pts**)

June 13, 2023.

1. (10 pts) Find the area of the region $\mathcal{R} = \{x > 0,\ y > 0,\ \sqrt{x} + \sqrt{y} < 1\}$.

   Answer.

   Let $u = \sqrt{x}$, $v = \sqrt{y}$ and $\widetilde{\mathcal{R}} = \{u > 0,\ v > 0,\ u + v < 1\}$.

$$\iint\limits_{\mathcal{R}} dx\, dy = \iint\limits_{\widetilde{\mathcal{R}}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv = \int_0^1 \int_0^{1-v} 4uv\, du\, dv = \int_0^1 2v(1-v)^2\, dv = \frac{1}{6}$$

2. (10 pts) Evaluate $I = \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx$.

   Answer.

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx\, dy = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2} r\, dr d\theta$$

   Therefore

$$I = \left(\int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2} r\, dr d\theta\right)^{\frac{1}{2}} = \left(\int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{1}{2}r^2} r\, dr\right)^{\frac{1}{2}} = \sqrt{2\pi}$$

3. (15 pts) Rewrite $\displaystyle\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dz\, dy\, dx$ in the order $dy\, dz\, dx$ and $dx\, dy\, dz$, respectively and evaluate the integral.

   Answer.

$$= \int_{-1}^{1}\int_0^{1-x^2}\int_{x^2}^{1-z} dy\, dz\, dx = \int_0^1 \int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz = \frac{8}{15}$$

4. (15 pts) Replace

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta$$

   by a triple integral in spherical coordinates and evaluate the integral.

   Answer.

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\rho=0}^{\frac{\sqrt{2}}{\cos\phi}} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta + \int_{\theta=0}^{2\pi}\int_{\phi=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\rho=0}^{2} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}}\int_{\rho=0}^{2} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta - \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{4}}\int_{\frac{\sqrt{2}}{\cos\phi}}^{2} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\rho=0}^{\sqrt{2}} \rho^2 \int_{\phi=0}^{\frac{\pi}{2}} \sin\phi \, d\phi \, d\rho \, d\theta + \int_{\theta=0}^{2\pi} \int_{\rho=\sqrt{2}}^{2} \rho^2 \int_{\phi=\cos^{-1}\left(\frac{\sqrt{2}}{\rho}\right)}^{\frac{\pi}{2}} \sin\phi \, d\phi \, d\rho \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} \rho^2 \int_{\phi=0}^{\frac{\pi}{2}} \sin\phi \, d\phi \, d\rho \, d\theta - \int_{\theta=0}^{2\pi} \int_{\rho=\sqrt{2}}^{2} \rho^2 \int_{\phi=0}^{\cos^{-1}\left(\frac{\sqrt{2}}{\rho}\right)} \sin\phi \, d\phi \, d\rho \, d\theta = \frac{10\sqrt{2}\pi}{3}$$

5. (15 pts) State and prove the Fundamental Theorem of Line Integrals.

   Answer.

   See p985-p986 of the textbook.

6. Let $\boldsymbol{F}(x, y) = (M(x, y), N(x, y))$. Suppose M, N and their first partial derivatives are all continuous on the $x - y$ plane and let $\mathcal{R} = \{x^2 + y^2 < 1, x > 0, y > 0\}$.

   (a) (8 pts) State Green's Theorem in both forms for $\boldsymbol{F}$ on $\mathcal{R}$.

   (b) (12 pts) Verify either one of the two forms on $\mathcal{R}$. That is, evaluate the line integral and the double integral and check that they are the same.

Ans:

(a) Let C be the boundary of $\mathcal{R}$.

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{n} \, ds = \oint_C M \, dy - N \, dx = \iint_{\mathcal{R}} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy,$$

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{T} \, ds = \oint_C M \, dx + N \, dy = \iint_{\mathcal{R}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy.$$

(b) (For example, in tangential form) Let $C = C_1 \cup C_2 \cup C_3$, where

$$\begin{aligned} C_1 &: \quad \boldsymbol{r}(t) = (t, 0), & 0 \le t \le 1, \\ C_2 &: \quad \boldsymbol{r}(t) = \left(\sqrt{1 - t^2}, t\right), & 0 \le t \le 1, \\ C_3 &: \quad \boldsymbol{r}(t) = (0, 1 - t), & 0 \le t \le 1. \end{aligned}$$

Then

$$\int_{C_1} M\,dy = \int_0^1 M(t,0)\cdot 0\,dt = 0,$$

$$\int_{C_2} M\,dy = \int_0^1 M(\sqrt{1-t^2},t)\cdot 1\,dt = \int_0^1 M(\sqrt{1-t^2},t)\,dt,$$

$$\int_{C_3} M\,dy = \int_0^1 M(0,1-t)\cdot(-1)\,dt = -\int_0^1 M(0,u)\,du,\ (u=1-t),$$

$$\int_{C_1} N\,dx = \int_0^1 N(t,0)\cdot 1\,dt = \int_0^1 N(t,0)\,dt.$$

$$\int_{C_2} N\,dx = \int_0^1 N(\sqrt{1-t^2},t)\cdot\frac{d\sqrt{1-t^2}}{dt}\,dt = -\int_0^1 N(v,\sqrt{1-v^2})\,dv,\ (v=\sqrt{1-t^2})$$

$$\int_{C_3} N\,dx = \int_0^1 N(0,1-t)\cdot 0\,dt = 0.$$

Hence,

$$\begin{aligned}
&\int_C M\,dy - N\,dx \\
&= \left(\int_{C_1} M\,dy + \int_{C_2} M\,dy + \int_{C_3} M\,dy\right) - \left(\int_{C_1} N\,dx + \int_{C_2} N\,dx + \int_{C_3} N\,dx\right) \\
&= \int_0^1 M(\sqrt{1-t^2},t)\,dt - \int_0^1 M(0,u)\,du + \int_0^1 N(v,\sqrt{1-v^2})\,dv - \int_0^1 N(t,0)\,dt.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\iint_{\mathcal{R}} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \\
&= \iint_{\mathcal{R}} \frac{\partial M}{\partial x}\,dx\,dy + \iint_{\mathcal{R}} \frac{\partial N}{\partial y}\,dx\,dy \\
&= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\partial M}{\partial x}\,dx\,dy + \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\partial N}{\partial y}\,dy\,dx \\
&= \int_0^1 \Big[M(x,y)\Big]_{x=0}^{x=\sqrt{1-y^2}}\,dy + \int_0^1 \Big[N(x,y)\Big]_{y=0}^{y=\sqrt{1-x^2}}\,dx \\
&= \int_0^1 \Big(M(\sqrt{1-y^2},y) - M(0,y)\Big)\,dy + \int_0^1 \Big(N(x,\sqrt{1-x^2}) - N(x,0)\Big)\,dx.
\end{aligned}$$

Hence,

$$\int\limits_C M\,dy - N\,dx = \iint\limits_{\mathcal{R}} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dx\,dy.$$

7. (15 pts) Evaluate $\oint_{\mathcal{C}} \boldsymbol{F}\cdot\boldsymbol{n}\;ds$, where $\boldsymbol{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ and $\mathcal{C} = \left\{\frac{x^2}{9} + \frac{y^2}{4} = 1\right\}$.

Answer.

The boundary integral is part of Green's Theorem in normal form. Therefore we first check that

$$M_x(x,y) + N_y(x,y) = 0, \quad (x,y) \neq (0,0).$$

We can apply Green's Theorem in normal form over the region $\mathcal{R}_a = \left\{x^2+y^2 > a^2, \frac{x^2}{9} + \frac{y^2}{4} < 1\right\}$ to get

$$0 = \iint\limits_{\mathcal{R}_a} M_x + N_y\,dA = \oint_{\mathcal{C}} \boldsymbol{F}\cdot\boldsymbol{n}\;ds - \oint_{\mathcal{C}_a} \boldsymbol{F}\cdot\boldsymbol{n}\;ds$$

where $\mathcal{C}_a = \{x^2+y^2 = a^2\}$ for any a, $0 < a < 2$. Here $\boldsymbol{n}$ on both $\mathcal{C}$ and $\mathcal{C}_a$ are pointing outward.

By direct calculation ($x(\theta) = a\cos\theta, y(\theta) = a\sin\theta$, $0 < \theta < 2\pi$),

$$\oint_{\mathcal{C}_a} \boldsymbol{F}\cdot\boldsymbol{n}\;ds = \int_0^{2\pi} (\frac{\cos\theta}{a}, \frac{\sin\theta}{a})\cdot(\cos\theta, \sin\theta)\;a\,d\theta = 2\pi$$

Therefore we conclude that

$$\oint_{\mathcal{C}} \boldsymbol{F}\cdot\boldsymbol{n}\;ds = \oint_{\mathcal{C}_a} \boldsymbol{F}\cdot\boldsymbol{n}\;ds = 2\pi$$