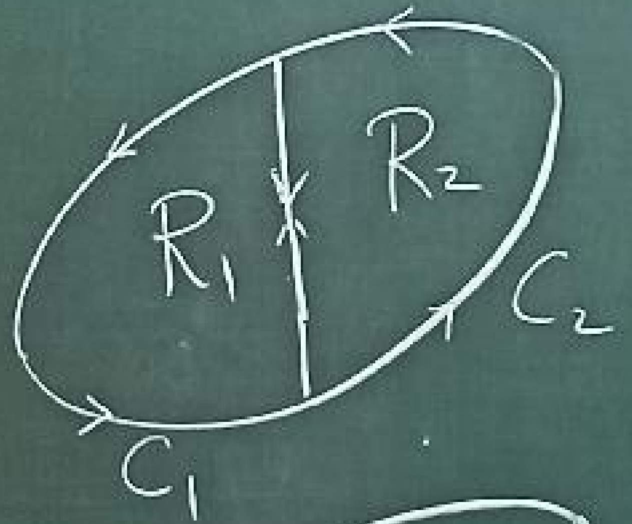
Pf of Green's Theorem (tangential form)

(i) $\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx+N\,dy = \iint_R (N_x - M_y)\,dA$

(pf for normal form is similar)

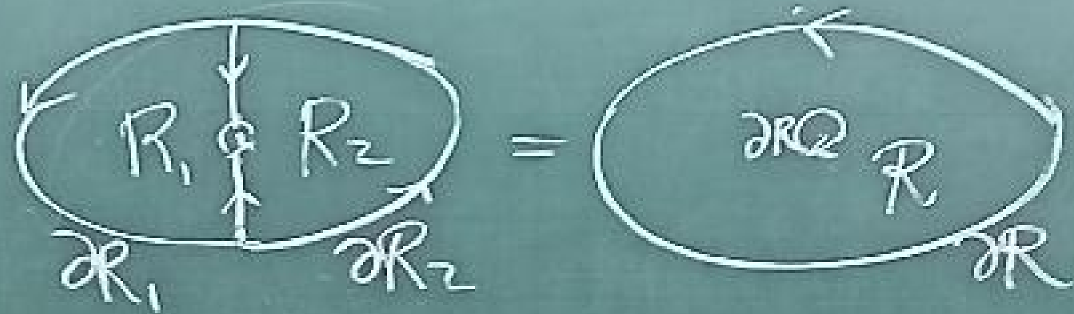


If (i) is true for (C_1,R_1), (C_2,R_2) then it is true for (C,R) where

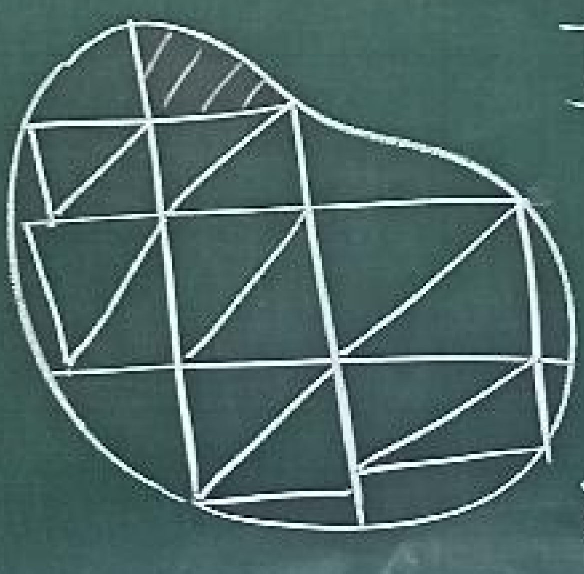
$C=C_1+C_2$

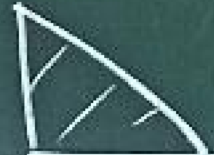
$R=R_1\cup R_2$

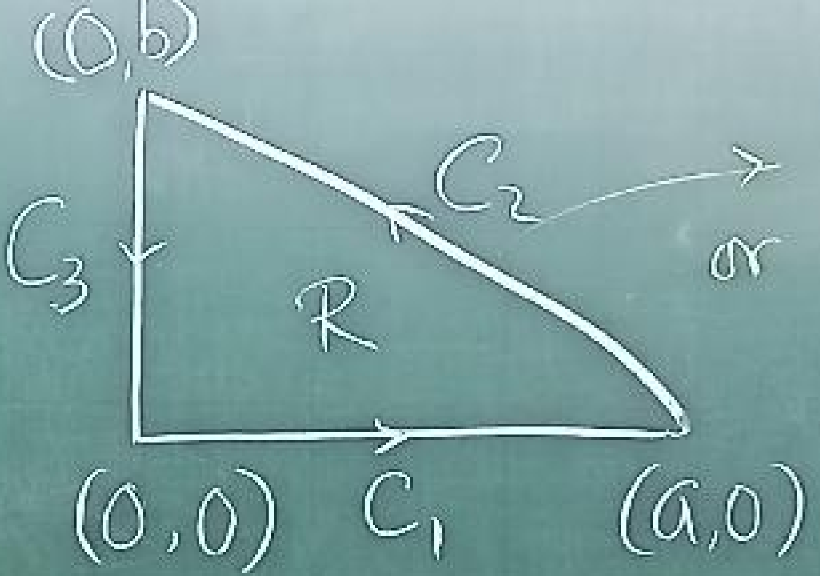
Rm in previous example



Similarly



In general,
it suffice to
verify R
of the form 



$y = f(x) \quad 0 \le x \le a$

or $x = g(y) \quad 0 \le y \le b$

$$C_2 = \{(x, f(x)),\ 0 \le x \le a\} = \{(g(y), y),\ 0 \le y \le b\}$$

Check tangential form, line integrals:

$$\int_{C_1} \underset{\vec{T}=(1,0)}{\overset{t=x}{=}} \int_{t=0}^{a} \vec{F}\cdot(1,0)\,dt = \int_0^a M(t,0)\,dt = \int_0^a M(x,0)\,dx \qquad (\mathrm{I})$$

$$\int_{C_3} \underset{\vec{T}=(0,-1)}{\overset{t=b-y}{=}} \int_{t=0}^{b} -N(0,b-t)\,dt = -\int_{y=b}^{0} N(0,y)(-dy)$$

$$\|$$

$$\text{or } \int_{C_3\downarrow} = -\int_{-C_3\uparrow} \underset{\vec{T}=(0,1)}{\overset{s=y}{=}} -\int_{s=0}^{b} N(0,s)\,ds = -\int_{y=0}^{b} N(0,y)\,dy \qquad (\mathrm{III})$$

$$\int_{C_2} \underset{t=y}{=} \int_{t=0}^{b} \left(M\frac{dx}{dt} + N\frac{dy}{dt} \right) dt = \int_0^b \Big(\underbrace{M(g(t),t)\,g'(t)}_{(\mathrm{II}_1)} + \underbrace{N(g(t),t)}_{(\mathrm{II}_2)} \Big) dt$$

$C_2 = \{ y=t,\ x=g(t),\ 0 \le t \le b \}$ $\quad \dot{x} = g'(t),\ \dot{y} = 1$

Double integrals $\iint_R (N_x - M_y)\, dA$

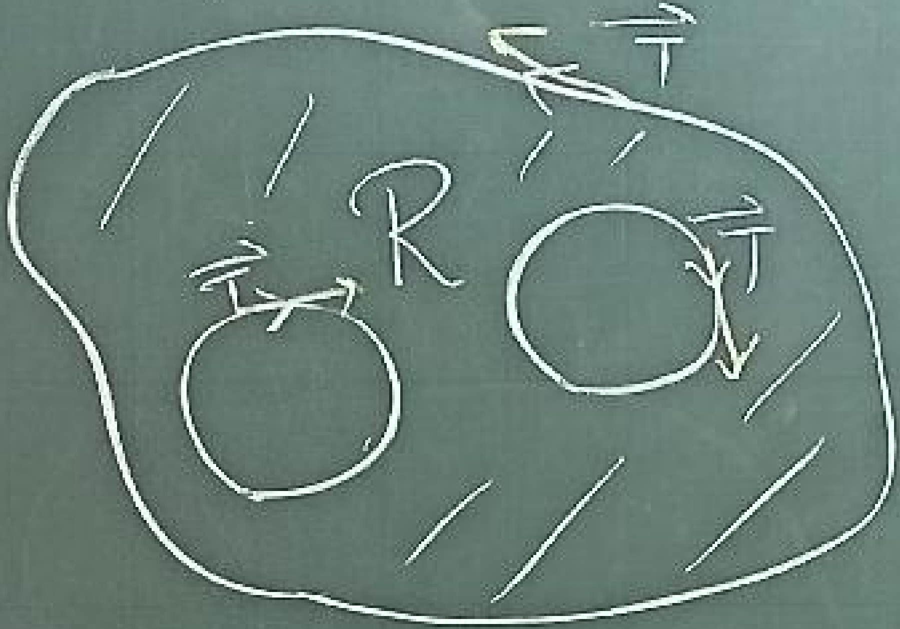
$$\iint_R N_x\, dA = \int_0^b \int_0^{g(y)} N_x\, dx\, dy = \int_0^b \Big(\underbrace{N(g(y),y)}_{(\mathrm{II}_2)} - \underbrace{N(0,y)}_{(\mathrm{III})} \Big) dy$$

$$-\iint_R M_y\, dA = -\int_0^a \int_{y=0}^{f(x)} M_y\, dy\, dx = -\int_0^a \Big(\underbrace{M(x,f(x))}_{(\mathrm{IV})} - \underbrace{M(x,0)}_{(\mathrm{I})} \Big) dx$$

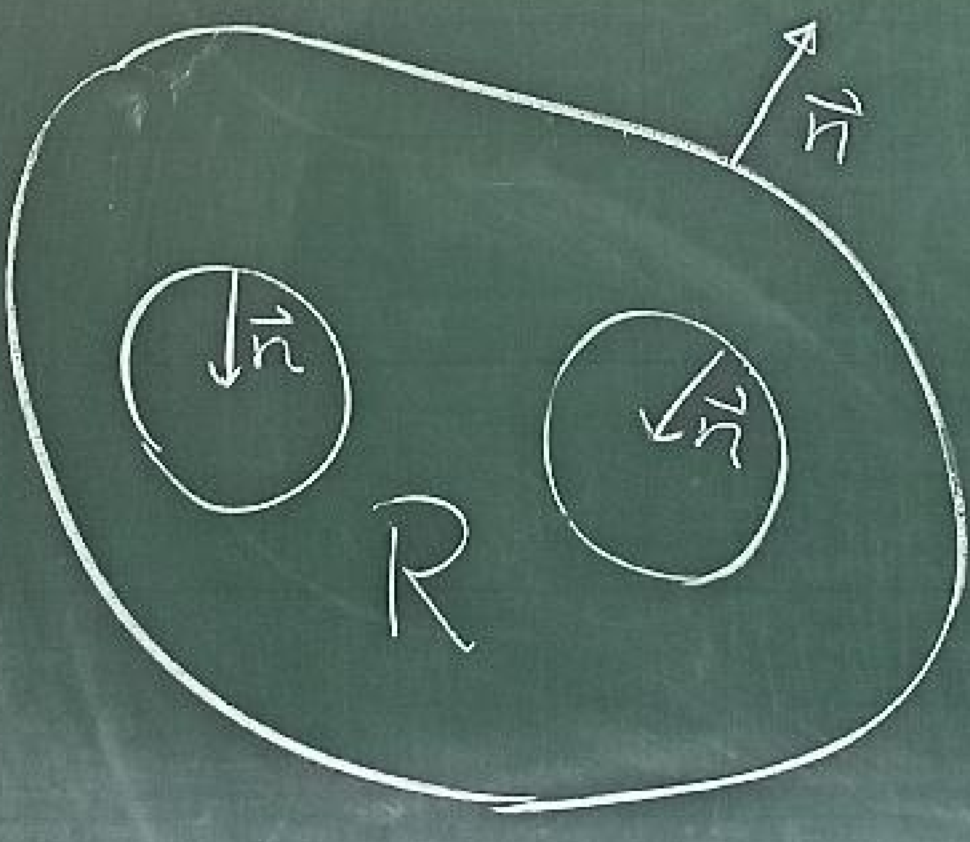
$$(\mathrm{IV}) = -\int_0^a M(x,f(x))\, dx \qquad \begin{aligned} y &= f(x) \\ x &= f^{-1}(y) \equiv g(y) \end{aligned}$$

$$= -\int_{y=b}^{0} \underbrace{M(g(y),y)\, g'(y)\, dy}_{=(\mathrm{II}_1)} \qquad dx = g'(y)\, dy \qquad \therefore \text{(i) verified}$$

Tangential form



Normal form



Rm For $\vec{F} = \frac{(-y, x)}{x^2+y^2}$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \begin{cases} 2\pi, & (0,0) \in R \\ 0, & (0,0) \notin R \end{cases}$$

R = inside of C

$$\left[\begin{array}{l} \text{Rm } \frac{(-y, x)}{x^2+y^2} = \nabla \tan^{-1}\left(\frac{y}{x}\right) \\ \text{But } \theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ is NOT} \\ \text{everywhere defined on } x^2+y^2 \neq 0 \end{array}\right]$$