Let $\vec{F} = (M, N, P)$

M, N, P and their first derivatives are cont. in D

$$\left\{\begin{array}{l} \displaystyle\int_A^B \vec{F}\cdot\vec{T}\,ds \text{ is path indep.} \\ \Leftrightarrow \displaystyle\oint_C \vec{F}\cdot\vec{T}\,ds = 0 \text{ for all closed curves } C \text{ in } D \\ \Leftrightarrow \vec{F} = \nabla f \end{array}\right.$$

$$\begin{array}{l} \Longrightarrow \\ \not\Longleftarrow^{(*)} \end{array} \begin{cases} M_y = N_x \\ N_z = P_y \\ P_x = M_z \end{cases}$$

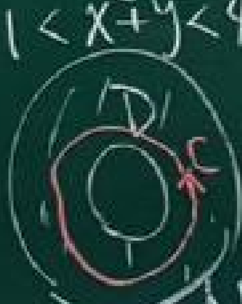
$\Leftarrow$ (*) holds if D is simply connected.

Def D is simply connected if every closed loop in D can be contracted to a point without leaving D

Eg 1 Simply connected?

$x^2+y^2<4$ $\qquad$ $1<x^2+y^2<4$

Yes



No

D	Simply connected?
$x^2+y^2+z^2<4$	Y
$1<x^2+y^2+z^2<4$	Y
$\mathbb{R}^3 \setminus \{(0,0,0)\}$	Y
$1<x^2+y^2<4$ in $\mathbb{R}^3$	No
$\mathbb{R}^3 \setminus z$-axis	No
 $(\sqrt{x^2+y^2}-2)^2+z^2<1$	No

If D is not simply connected

$\xLeftarrow{(*)}$ may or may not be true, depending on $\vec{F}$.

Eg 2 $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$

Is $\vec{F}$ conservative ?

$\underline{\text{Sol}}$: $M_y = N_x = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\vec{F}$ is defined on $\underbrace{\mathbb{R}^3 \setminus z\text{-axis}}_{(\star)}$

The natural domain $(\star)$ is not simply connected.

Take C: $\vec{r}(t) = (r\cos t, r\sin t, 0)$

$r > 0$ fixed. $0 \le t \le 2\pi$

ed

$$\oint_C \vec{F} \cdot \vec{T}\, ds \begin{cases} = 0 \Leftrightarrow \Longleftarrow^{(*)} \\ \neq 0 \Leftrightarrow \not\Longleftarrow^{(*)} \end{cases}$$

$$\vec{T} = \frac{(-r\sin t,\ r\cos t,\ 0)}{r} = (-\sin t,\ \cos t,\ 0)$$

$$ds = \left|\frac{d\vec{r}}{dt}\right| dt = r\,dt$$

$$\oint_C = \int_0^{2\pi} \left(\frac{-r\sin t}{r^2}, \frac{r\cos t}{r^2}, 0\right)\cdot(-\sin t,\ \cos t,\ 0)\, r\,dt$$

$$= \int_0^{2\pi} 1\,dt = 2\pi \neq 0$$

$\therefore \vec{F}$ is not conservative

Eg3: $\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 0\right)$

$N_y = M_x$ (check)

Natural domain $= \mathbb{R}^3 \setminus z$-axis

Is $\vec{F}$ conservative? (hw)

Green's Theorem

C: a simple closed curve

$\vec{F} = (M, N)$, M, N and their first derivatives are cont. in R = interior of C (i.e., $C = \partial R$)

Then

(i) $$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\, dA$$

(normal form) $\vec{n}\,ds = (dy, -dx) = \left(\frac{dy}{dt}, -\frac{dx}{dt}\right) dt$

(ii) $$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$

(tangential form) $\vec{T}\,ds = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) dt$

Rm $\vec{F} = (M, N) \longleftrightarrow \vec{G} = (N, -M)$

normal form for $\vec{F}$ = tangential form for $\vec{G}$
(tangential) (normal)

Rm They are special cases of

(Gauss) $\iiint_D \nabla \cdot \vec{F}\, dV = \iint_{\partial D} \vec{F} \cdot \vec{n}\, d\sigma$
$(\operatorname{div} \vec{F})$

(Stokes') $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{T}\, ds$

$d\sigma$: surface integral
∂D: boundary of D (domain)
∂S: boundary of S (surface)

dt

whee

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} \left(= (\partial_x, \partial_y, \partial_z)\cdot(F_1, F_2, F_3)\right)$$

$$= \partial_x F_1 + \partial_y F_2 + \partial_z F_3$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} \left(= (\partial_x, \partial_y, \partial_z)\times(F_1, F_2, F_3)\right)$$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\partial_y F_3 - \partial_z F_2,\ \partial_z F_1 - \partial_x F_3,\ \partial_x F_2 - \partial_y F_1\right)$$

Special case of Stokes': Take $S = R$. $\partial S = C$

$$\vec{F} = (\overset{F_1}{M}, \overset{F_2}{N}, 0),\quad \vec{n} = (0,0,1) \Leftrightarrow C = \circlearrowleft$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma = \iint_R \underbrace{\partial_x F_2 - \partial_y F_1}_{= N_x - M_y}\, dA = \oint_C \vec{F}\cdot\vec{T}\, ds$$

Eg3 Verify both forms of Green's Thm for

$\vec{F}(x,y) = (x-y, x)$ and

$C: \vec{r}(t) = (\cos t, \sin t),\ 0 \le t \le 2\pi$

<u>Sol</u> normal form

$$\iint_{x^2+y^2\le 1} M_x + N_y \, dA = \oint_C \vec{F} \cdot \vec{n} \, ds$$

$$LHS = \iint_{x^2+y^2\le 1} 1 \, dA = \pi$$

$$RHS = \oint_C \vec{F} \cdot \left(\frac{dy}{dt}, -\frac{dx}{dt}\right) dt = \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} dt = \pi$$

Tangential form

$$\iint_{x^2+y^2\le 1} (N_x - M_y)\,dA = \oint_C M\,dx + N\,dy$$

$$\text{LHS} = \iint_{x^2+y^2\le 1} 2\,dA = 2\pi$$

$$\text{RHS} = \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_0^{2\pi} -\cos t \sin t\,dt + \int_0^{2\pi} 1\,dt$$

$$= 2\pi.$$

Eg4 Evaluate $\oint_C \vec{F}\cdot\vec{T}\,ds$

where $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = (M, N)$

$C: x^2+y^2 = a^2$

Sol: Component test

$M_y - N_x = 0$

Method 1: Green's Thm (tangential)

$\iint_{x^2+y^2\le a^2} 0\,dA = 0$ (Wrong!)

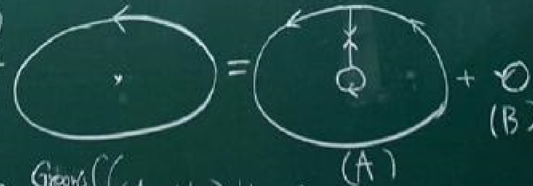
Method 2: $x = a\cos t,\ y = a\sin t$

$\text{Ans} = \int_0^{2\pi} \begin{pmatrix} \frac{-\sin t}{a} \\ \frac{\cos t}{a} \end{pmatrix} \cdot \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix} dt = 2\pi$ (correct)

Note: M, N and their first derivatives are not cont. in $x^2+y^2 \le a^2$!

Eg 5 Evaluate $\oint_C \vec{F}\cdot\vec{T}\,ds$ where $\vec{F}=\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ and $C: \frac{x^2}{4}+y^2=1$

Sol



(A): $\overset{\text{Green's}}{=} \iint (M_y - N_x)\,dA = 0$ Ans: $=2\pi$

(B) $\overset{\text{HW}}{=} \oint_{x^2+y^2=a^2,\ 0<a<1} M\,dx+N\,dy = 2\pi$