Eg1: $\vec{F}=(x-y,\,y)$, $C=\{x^2+y^2=1\}$

$$\oint_C \vec{F}\cdot\vec{T}\,ds=? \qquad \oint_C \vec{F}\cdot\vec{n}\,ds=?$$

$$(\vec{F}\cdot d\vec{r})$$

Sol: C: $x(t)=\cos t$, $y(t)=\sin t$, $0\le t\le 2\pi$

$$\dot{x}(t)=-\sin t,\quad \dot{y}(t)=\cos t$$

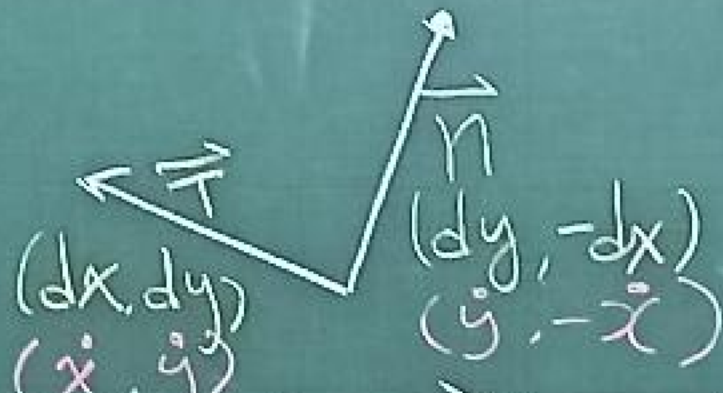
(i) $$\oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\underbrace{(F_1\cdot\dot{x}+F_2\cdot\dot{y})\,dt}_{F_1dx+F_2dy}$$

$$=\int_0^{2\pi}\big((\cos t-\sin t)(-\sin t)+\sin t\cos t\big)\,dt$$

$$=\int_0^{2\pi}\sin^2 t\,dt=\pi$$

(ii) $\vec{T} = (-\sin t, \cos t)$

$\vec{n} = (\cos t, \sin t)$



$$d\vec{r} = \vec{T}\,ds = (dx, dy)$$

$$\vec{n}\,ds = (dy, -dx)$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} F_1\,dy - F_2\,dx$$

$$= \int_0^{2\pi} \underline{(F_1\dot{y} - F_2\dot{x})\,dt}$$

$$= \int_0^{2\pi} (\cos t - \sin t)\cos t - \sin t(-\sin t)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 2\pi$$

Fundamental Theorem of Line Integrals

Thm 1: If f is diff. in D, ∇f is cont. in D and C is any curve in D from point A to point B, then

$$\int_C \underbrace{\nabla f \cdot \vec{T}\,ds}_{\nabla f \cdot d\vec{r}} = f(B) - f(A)$$

pf: Let $\vec{r}(t)$, $0 \le t \le 1$ be a parametrization of C, $\vec{r}(0) = A$, $\vec{r}(1) = B$.

$$\Rightarrow \int_C \nabla f \cdot \vec{T}\,ds = \int_0^1 \left(f_x(x(t), y(t))\,\dot{x} + f_y(x(t), y(t))\,\dot{y}(t) \right) dt$$

$$= \int_0^1 \frac{d}{dt} f(x(t), y(t))\,dt = f(x(t), y(t))\Big|_{t=0}^{1} = f(B) - f(A)$$

Def: $\vec{F}$ is conservative in D

if for any curve $C \subseteq D$

$\int_C \vec{F} \cdot \vec{T}\, ds$ only depends on

the end points of C

(i.e. path-independent)

Thm says: if f is diff.

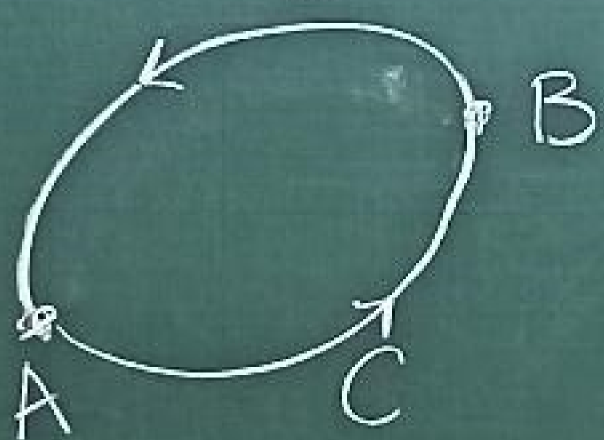
∇f is cont. then

$$\vec{F} = \nabla f \Rightarrow \vec{F} \text{ is conservative}$$

Note: "$\Leftarrow$" is also true

Rm1: $\vec{F}$ is conservative in D
(= Theorem 3)

$\Leftrightarrow \int_C \vec{F}\cdot\vec{T}\,ds = 0$ for any closed curve $C \subseteq D$



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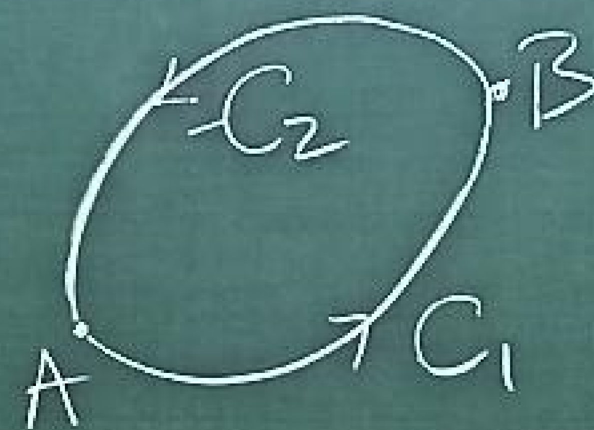
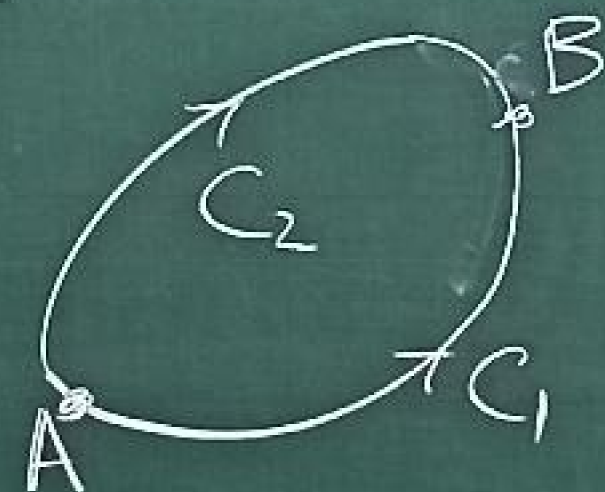


$$\int_C = \int_{C_1} + \int_{-C_2}$$



$$= \int_{C_1} - \int_{C_2}$$

$$\therefore \int_C = 0 \Leftrightarrow \int_{C_1} = \int_{C_2}$$

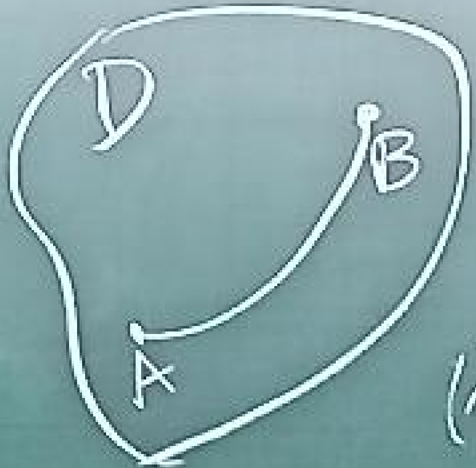
Thm 2: Let D be an open connected domain, $\vec{F}: D \to \mathbb{R}^3$ cont.

Then "$\vec{F}$ is conservative in D"

$\Longleftrightarrow$ "$\vec{F} = \nabla f$ for some diff. function f in D".

Pf: "$\Leftarrow$": previous Theorem (Fundamental Theorem of line integrals)

"$\Rightarrow$": We construct this f explicitly as follows.



Take any point $A \in D$

(i) Define $f(A) = 0$

(ii) For any $B \in D$

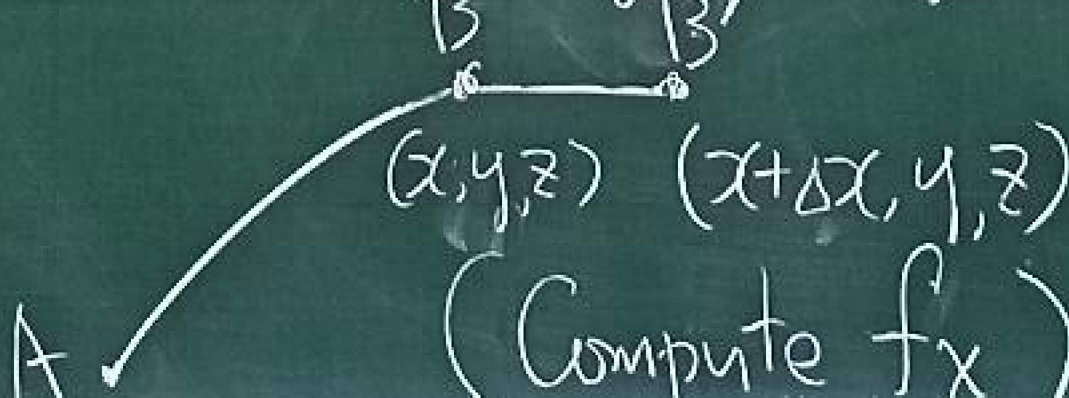
define $f(B) = \int_C \vec{F} \cdot \vec{T}\, ds$

where C is <u>any</u> curve from A to B

$\Rightarrow$ f is defined everywhere in D.

(Note $f(B)$ is independent of C)

Question: Why $\nabla f = \vec{F}$?



(Compute f_x)

$$f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_{A\to B\to B'} \vec{F}\cdot\vec{T}\,ds - \int_{A\to B} \vec{F}\cdot\vec{T}\,ds}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_{B\to B'} \vec{F}\cdot\vec{T}\,ds}{\Delta x}$$

$$\left(B \to B' : \vec{T} = (1,0,0),\ \vec{F}\cdot\vec{T} = F_1\right)$$

$$= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} F_1(t,y,z)\,dt}{\Delta x} = F_1(x,y,z)$$

i.e. $\overline{BB'} = \{(t,y,z),\ x \le t \le x+\Delta x\}$

$\therefore f_x(x,y,z) = F_1(x,y,z)$ Similarly

$f_y = F_2,\ f_z = F_3 \ \therefore \nabla f = \vec{F}$

Given $\vec{F}$, how to determine whether $\vec{F}$ is conservative in D?

Note: (1) If $\vec{F}(x,y) = (M(x,y), N(x,y))$

$\vec{F}$ is conservative $\underset{(\not\Leftarrow)}{\Longrightarrow}$ $M_y = N_x$ $\cancel{((M,N) \text{ is } \sout{\text{exact}} \text{ closed})}$

(component test)

$\left(\begin{array}{c} M = f_x, N = f_y \\ \text{for some } f \end{array} \right)$ see hw 13, problem 2 $\quad (= f_{xy} = f_{yx})$

(2) If $\vec{F}(x,y,z) = (M(x,y,z), N, P)$ is conservative,

$$\begin{array}{c} \Longrightarrow \\ \not\Leftarrow \end{array} \begin{cases} M_y = N_x \\ N_z = P_y \\ M_z = P_x \end{cases} (*) \quad \cancel{\left(\begin{array}{c} \text{Def: } (M,N,P) \\ \text{is } \sout{\text{exact}} \text{ closed} \end{array} \right)}$$

see hw 13, problem 3 (component test)

Rm2: If D is simply connected (next lecture)
then "$\Leftarrow$" is also true.

Eg: $\vec{F} = (e^x \cos y + yz, xz - e^x \sin y, xy + z)$

Show that $\vec{F}$ is conservative.
(find the potential f)

Sol: Check $(*)$ first

$$M_y = -e^x \sin y + z = N_x$$
$$N_z = x \qquad = P_y$$
$$P_x = y \qquad = M_z$$

$\Rightarrow f$ may probably exist!
($D = \mathbb{R}^3$ = simply connected $\Rightarrow f$ exists)
(from Rm 2 above)

Try to find f!

$$f_x = e^x \cos y + yz$$

$$\Rightarrow f = \int e^x \cos y + yz \, dx$$

$$= e^x \cos y + xyz + C_1(y,z)$$

Similarly $f = \int f_y \, dy$

$$f = xyz + e^x \cos y + C_2(x,z)$$

and $f = xyz + \frac{z^2}{2} + C_3(x,y)$

$\therefore$ take $f(x,y,z)$

$= xyz + e^x \cos y + \frac{z^2}{2}$ will do