Line integrals

Eg 1 Evaluate $\int_C f(x,y,z)\, ds$

where $f = x - 3y^2 + z$

C = line segment <u>between</u> $(0,0,0)$ and $(1,1,1)$.

<u>Sol</u>: <u>Step 1</u>: Find $\vec{r}(t)$ for C

such as $\vec{r}(t) = (t,t,t)$, $0 \le t \le 1$

<u>Step 2</u> $ds = |\dot{\vec{r}}(t)|\, dt = |(1,1,1)|\, dt$

<u>Step 3</u>: Ans $= \int_{t=0}^{1} (t - 3t^2 + t)\sqrt{3}\, dt = 0$

Rm $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

Rm: the value of $I = \int_C f(x,y,z)\,ds$.
is independent of the parameter
$\vec{r}(t)$. For example

$$\vec{r}(t) = (t^2, t^2, t^2),\ 0 \le t \le 1$$

or $\vec{r}(t) = (1-t, 1-t, 1-t),\ 0 \le t \le 1$

all give the same value of
I as long as $\vec{r}(t)$ is
a correct parametrization
for C.

Eg2: $f(x,y,z) = x - 3y^2 + z$

$C = C_1 \cup C_2$ (two line segments)

$C_1 = \overline{(0,0,0)\ (1,1,0)}$

$C_2 = \overline{(1,1,0)\ (1,1,1)}$

<u>Sol</u>: $C_1: \vec{r_1}(t) = (t, t, 0),\ 0 \le t \le 1$

$C_2: \vec{r_2}(t) = (1, 1, t),\ 0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2}$$

$$= \int_0^1 (t - 3t^2)\underbrace{\sqrt{2}}_{\left|\frac{d\vec{r_1}}{dt}\right|}\,dt + \int_0^1 (-2+t)\cdot \underbrace{1}_{\left|\frac{d\vec{r_2}}{dt}\right|}\cdot dt$$

$$= \frac{-\sqrt{2}}{2} + \frac{-3}{2}$$

(Compare with Eg1: same f, different C (same end points) $\Rightarrow$ different answers)

Related integral

(2) $\int_C \vec{F}\cdot\vec{T}\,ds$

$\vec{F}: D \longrightarrow \mathbb{R}^3$

$(x,y,z) \quad (F_1(x,y,z), F_2(\cdots), F_3(\cdots))$

C: a smooth curve with prescribed orientation(指向)

$\vec{r}(t)$: parametrization of C with "direction of increasing t"

= "orientation of C"

$\vec{T} = \dot{\vec{r}}(t) / |\dot{\vec{r}}(t)| \Rightarrow \vec{T}\,ds = \dot{\vec{r}}(t)\,dt$

Eg 3 Evaluate $\int_C \vec{F}\cdot\vec{T}\,ds$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

where $\vec{F} = (z, xy, -y^2)$

$C: \vec{r}(t) = (t^2, t, \sqrt{t}),\ 0 \le t \le 1$

Sol. $\dot{\vec{r}}(t) = \left(2t, 1, \frac{1}{2\sqrt{t}}\right)$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=0}^{1} \vec{F}\cdot\frac{\dot{\vec{r}}(t)}{|\dot{\vec{r}}(t)|}\,|\dot{\vec{r}}(t)|\,dt$$

$$= \int_0^1 \underbrace{(\sqrt{t}, t^3, -t^2)}_{\vec{F}}\cdot\underbrace{\left(2t, 1, \frac{1}{2\sqrt{t}}\right)}_{\dot{\vec{r}}(t)}\,dt$$

$$= \int_0^1 \left(2t^{\frac{3}{2}} + t^3 - \frac{t^{\frac{3}{2}}}{2}\right)dt$$

$$= \frac{3}{2}\cdot\frac{2}{5} + \frac{1}{4} = \frac{17}{20}$$

Rm If C is a simple (does not intersect itself) closed curve, we use

$$\oint_C \vec{F}\cdot\vec{T}\,ds$$

to specify the orientation of C.

Related integral

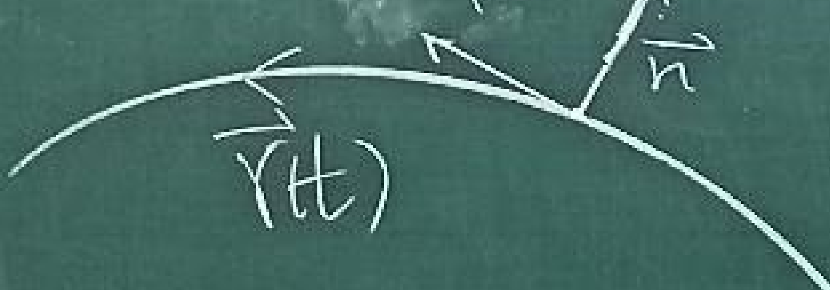
(3) $\int_C \vec{F}\cdot\vec{n}\,ds$ (flux)

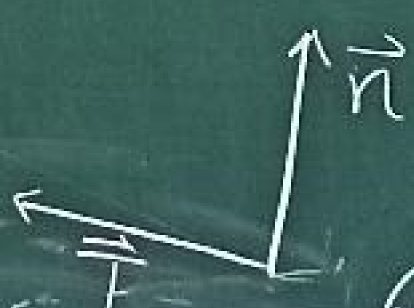
C: simple closed curve in a plane

$\vec{n}$: outward unit normal

To compute the flux, we need to compute $\vec{n}$ from $\vec{r}(t)$. For example, if $\vec{r}(t)$ is counterclockwise along C,

$$\vec{T} = \dot{\vec{r}}(t) / |\vec{r}(t)| = \frac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$





$$\Rightarrow \quad n_1 = T_2$$
$$n_2 = -T_1$$

$$\therefore \vec{n}\, ds = \frac{(\dot{y}(t), -\dot{x}(t))}{\sqrt{\dot{y}^2 + \dot{x}^2}} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = (\dot{y}(t), -\dot{x}(t))\, dt$$

Eg4. Evaluate $\int_C \vec{F} \cdot \vec{n}\, ds$

where $\vec{F} = (x-y, y)$, $C: x^2+y^2=1$

<u>Sol</u>. <u>Step 1</u>: Take $\vec{r}(t) = (\cos t, \sin t)$

$0 \leq t \leq 2\pi$

<u>Step 2</u>: $\dot{\vec{r}}(t) = (-\sin t, \cos t)$

Step 3: $\vec{F} \cdot \vec{n}\, ds = (F_1, F_2) \cdot (\dot{y}, -\dot{x})\, dt$

$$\text{Ans} = \int_0^{2\pi} (\cos t - \sin t, \sin t) \cdot (\cos t, \sin t)\, dt$$

$$= \int_0^{2\pi} \cos^2 t + \sin^2 t - \cos t \sin t\, dt$$

$$= 2\pi$$