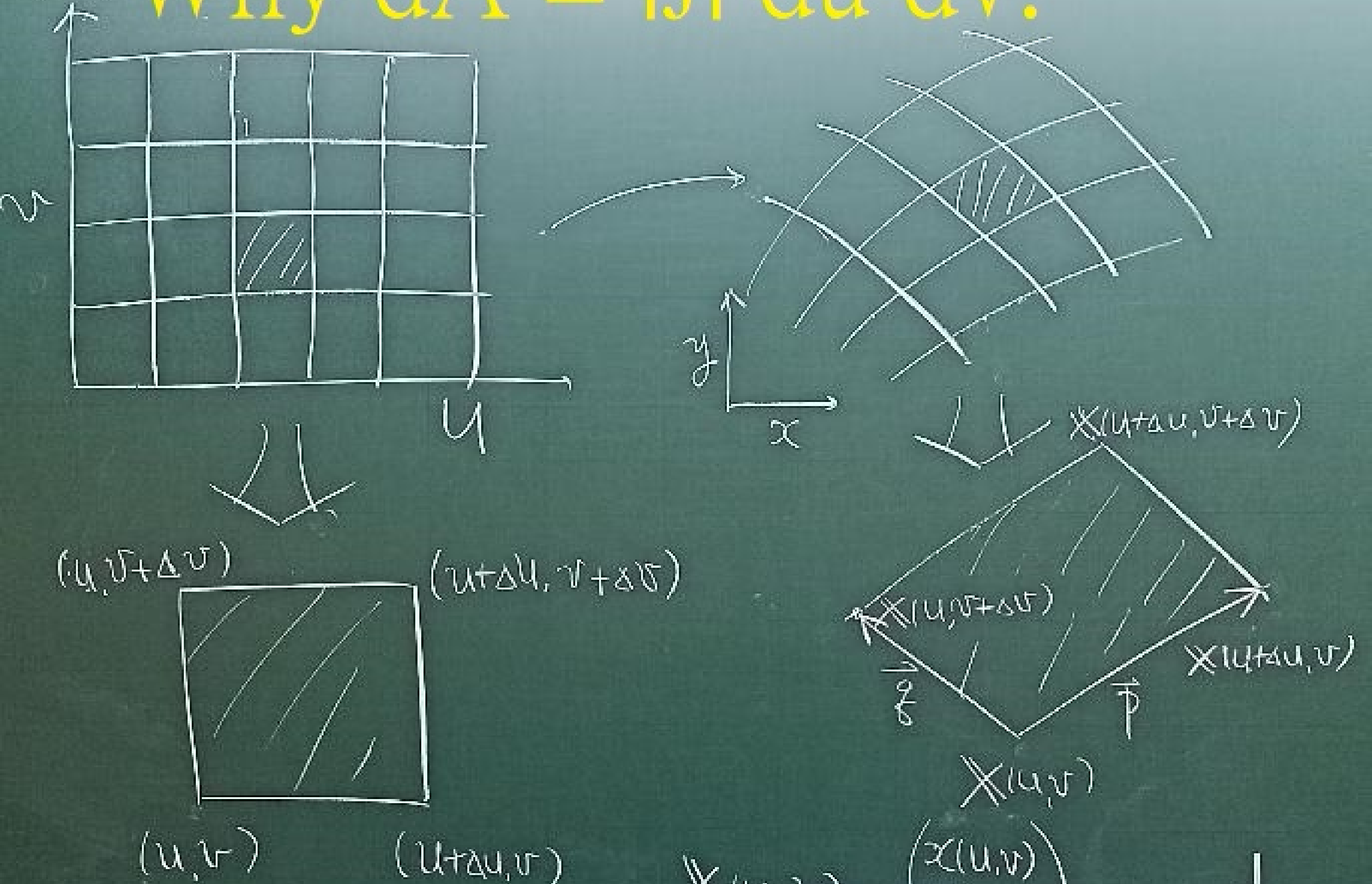
Why dA = |J| du dv:



$$\mathbb{X}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}, \quad \cdots \text{ etc.}$$

$$\mathbb{X}(u+\Delta u,v) = \begin{pmatrix} x(u+\Delta u,v) \\ y(u+\Delta u,v) \end{pmatrix}$$

$$\Delta A \cong |\vec{p} \times \vec{q}|$$

$$= |(\mathbb{X}(u+\Delta u, v) - \mathbb{X}(u,v)) \times (\mathbb{X}(u, v+\Delta v) - \mathbb{X}(u,v))|$$

$$= \left| \frac{\partial \mathbb{X}}{\partial u}(u+c_1\Delta u, v)\Delta u \times \frac{\partial \mathbb{X}}{\partial v}(u, v+c_2\Delta v)\Delta v \right|$$

$$= \left| \det\left(\frac{\partial \mathbb{X}}{\partial u} \downarrow , \frac{\partial \mathbb{X}}{\partial v} \downarrow \right) \right| \Delta u \Delta v$$

$$\left(\underline{\text{Note}} \quad \mathbb{X}_u = \begin{pmatrix} x_u \\ y_u \end{pmatrix} \quad \mathbb{X}_v = \begin{pmatrix} x_v \\ y_u \end{pmatrix} \right)$$

Note: $(a, b, 0) \times (c, d, 0)$

$$= \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix}$$

$$\left(\therefore \text{ We write } (a,b) \times (c,d) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

$$= \left(0, 0, \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$\Rightarrow |(a,b,0) \times (c,d,0)| = \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

$$\therefore \Delta A = |J| \Delta u \Delta v$$

$$J = \det\left(\frac{\partial \vec{x}}{\partial u} \cdot \frac{\partial \vec{x}}{\partial v}\right)$$

$$= \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Back to Eg1:

$$A = \int_{u=1}^{2}\int_{v=1}^{2} dA \qquad u = xy \quad v = \frac{y}{x}$$

$$= \int_1^2\int_1^2 |J| \, du\, dv \qquad x = \sqrt{\frac{u}{v}} \quad y = \sqrt{uv}$$

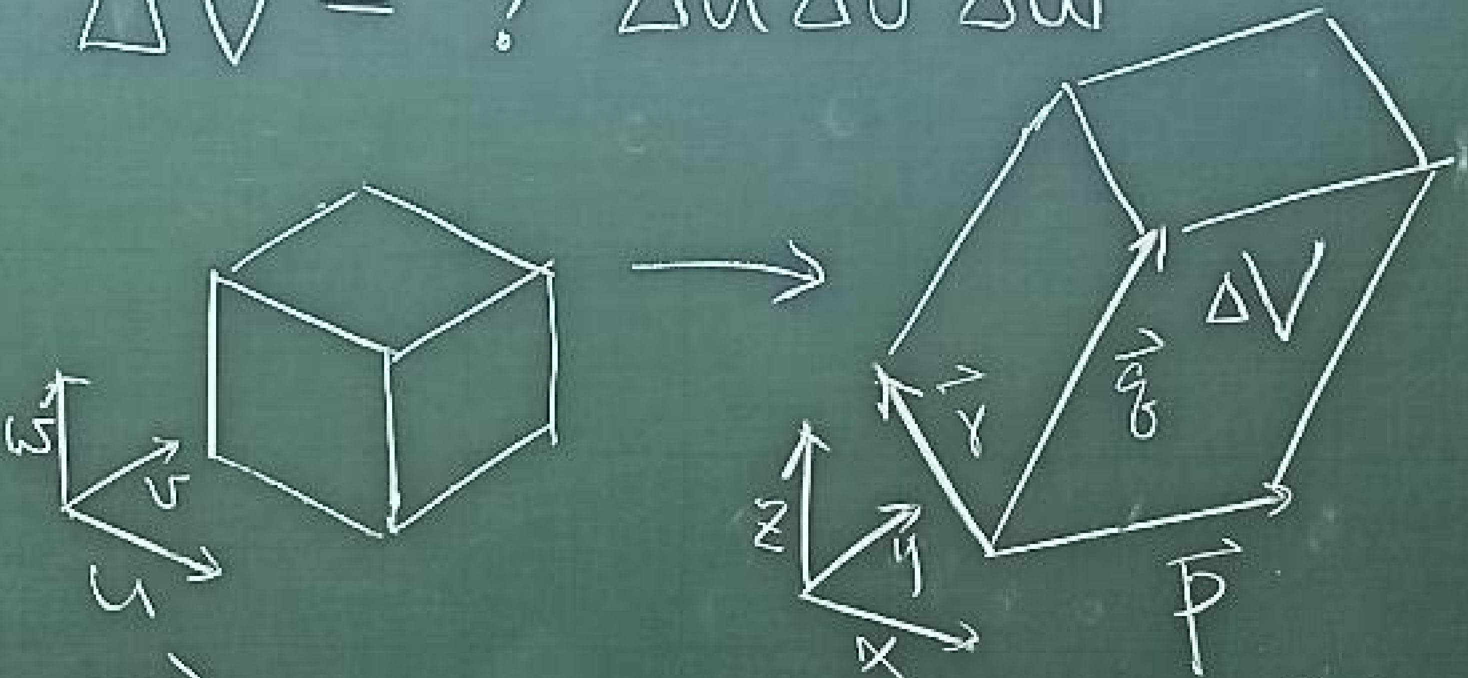
$$= \int_1^2\int_1^2 \left|\det\begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{uv}} & \frac{-1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix}\right| du\, dv$$

$$= \int_1^2\int_1^2 \left|\frac{1}{2v}\right| du\, dv = \ln 2/2$$

For Triple Integrals

$$u(x,y,z),\ v(x,y,z),\ w(x,y,z)$$

$$\Delta V = ?\ \Delta u \Delta v \Delta w$$



$$\vec{p} = \mathbb{X}(u+\Delta u, v, w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_u \Delta u$$

$$\vec{q} = \mathbb{X}(u, v+\Delta v, w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_v \Delta v$$

$$\vec{r} = \mathbb{X}(u, v, w+\Delta w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_w \Delta w$$

$$\Delta V = \left| \vec{p} \times \vec{q} \cdot \vec{r} \right| = \left| \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \right| \Delta u \Delta v \Delta w$$

$$= |J| \Delta u \Delta v \Delta w$$

Summary

Step 1 Find suitable

$u(x,y,z), v(x,y,z), w(x,y,z)$

for the problem.

<u>Step 2</u> Solve for $x(u,v,w)$

$y(u,v,w), z(u,v,w)$

Step 3.

$$J = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

$$\Delta V = |J| \Delta u \Delta v \Delta w$$

Eg 2:

Evaluate $\displaystyle\int_{z=0}^{3}\int_{y=0}^{4}\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$

using $u=x-\frac{y}{2}$, $v=\frac{y}{2}$, $w=\frac{z}{3}$

Sol

$x=u+v$, $y=2v$, $z=3w$

dw: $\displaystyle\int_{z=0}^{3}=\int_{w=0}^{1}$, dv: $\displaystyle\int_{y=0}^{4}=\int_{v=0}^{2}$

du: $\displaystyle\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}$ $\quad x=\frac{y}{2}\Leftrightarrow u=0$

$x=\frac{y}{2}+1\Leftrightarrow u=1$

$$J=\det\begin{pmatrix}x_u & x_v & x_w\\ y_u & y_v & y_w\\ z_u & z_v & z_w\end{pmatrix}=\det\begin{pmatrix}1&1&0\\0&2&0\\0&0&3\end{pmatrix}$$

$$=6$$

$$I = \int_{w=0}^{1}\int_{v=0}^{2}\int_{u=0}^{1}(u+w)\,\underline{6\,du\,dv\,dw}$$

$$= 6\left(\iiint u\,du\,dv\,dw + \iiint w\,du\,dv\,dw\right)$$

$$= 6\left(\int_{w=0}^{1}\int_{v=0}^{2}\frac{u^2}{2}\Big|_0^1\,dv\,dw + \int_{v=0}^{2}\int_{u=0}^{1}\left(\frac{w^2}{2}\right)\Big|_0^1\,du\,dv\right)$$

$$= 6\left(\frac{1}{2}\cdot 2\cdot 1 + \frac{1}{2}\cdot 1\cdot 2\right)$$

$$= 12$$

Eg 3: Spherical Coordinate

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$J = \det\begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix}$$

$$= \det\begin{pmatrix} S_\phi C_\theta & \rho C_\phi C_\theta & -\rho S_\phi S_\theta \\ S_\phi S_\theta & \rho C_\phi S_\theta & \rho S_\phi C_\theta \\ C_\phi & -\rho S_\phi & 0 \end{pmatrix} = \rho^2\sin\phi$$

(check!)

Line integrals

Goal $\displaystyle\int_{\vec{p}}^{\vec{q}} \nabla f(\vec{r})\cdot \underbrace{\vec{\tau}\, ds}_{d\vec{r}} = f(\vec{q}) - f(\vec{p})$

$(\vec{p}, \vec{q} \in \mathbb{R}^3 \quad f : \mathbb{R}^3 \longrightarrow \mathbb{R})$

We start with a related integral

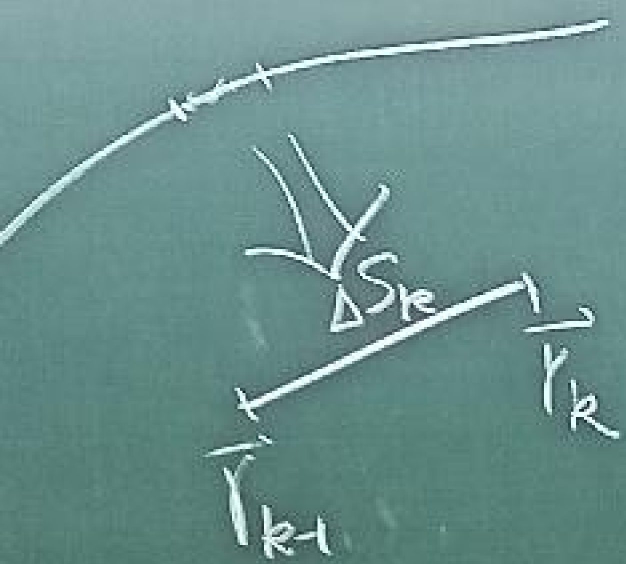
$$\int_C f(x,y,z)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta S_k$$

$D \subseteq \mathbb{R}^3$: domain of definition of f

$C \subseteq D$: a smooth curve.

$$C = \left\{ \vec{r}(t),\ a \le t \le b \right\}$$

$$= \left\{ \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},\ a \le t \le b \right\}$$



$$\vec{r}_k = \vec{r}(t_k) \qquad \Delta s_k = \left| \vec{r}_k - \vec{r}_{k-1} \right|$$

$$\Delta t_k = t_k - t_{k-1} \qquad = \frac{\left| \vec{r}_k - \vec{r}_{k-1} \right|}{\Delta t_k} \Delta t_k$$

$$\therefore\ ds = \left| \frac{d\vec{r}(t)}{dt} \right| dt$$