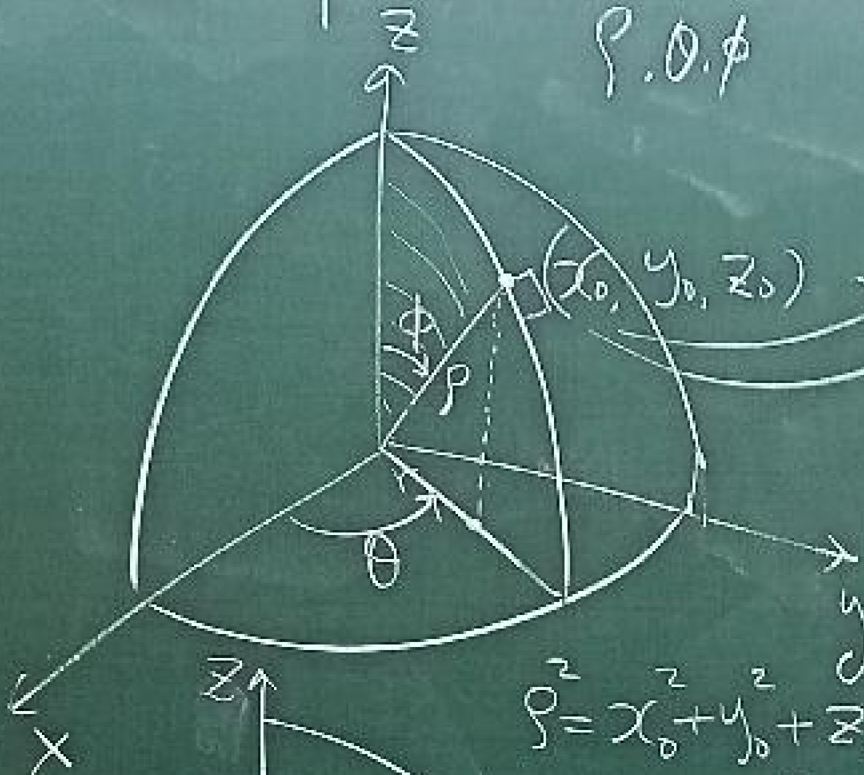
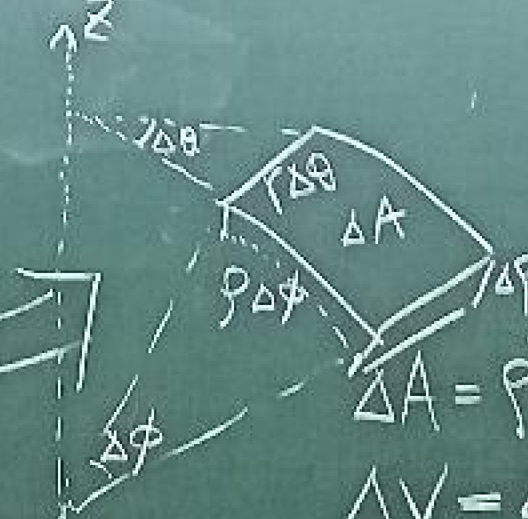
II Spherical Coordinates

ρ, θ, ϕ

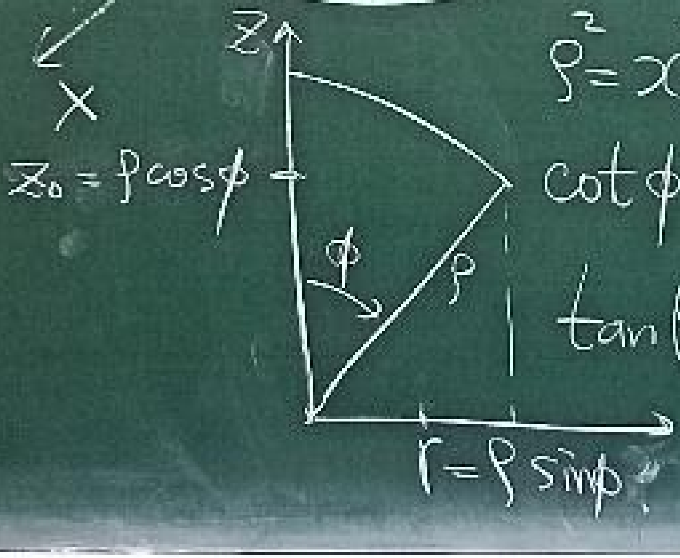




$$\Delta A = \rho^2 \sin\phi \,\Delta\theta\, \Delta\phi$$

$$\Delta V = \Delta A \,\Delta\rho = \rho^2 \sin\phi \,\Delta\rho\, \Delta\theta\, \Delta\phi$$

$$\rho^2 = x_0^2 + y_0^2 + z_0^2$$



$$\cot\phi = \frac{z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{z_0}{r}$$

$$\tan\theta = \frac{y_0}{x_0}$$

$$x_0 = r\cos\theta = \rho\sin\phi\cos\theta$$

$$y_0 = r\sin\theta = \rho\sin\phi\sin\theta$$

$$z_0 = \rho\cos\phi$$

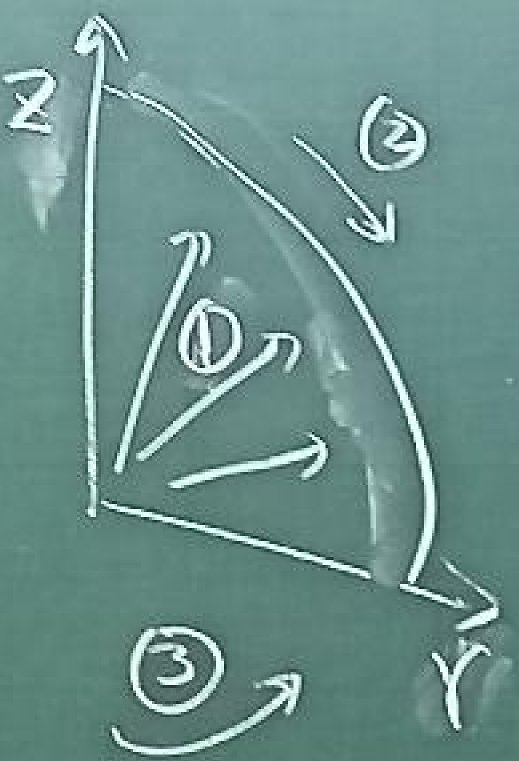
Cross Sections for various order of integration

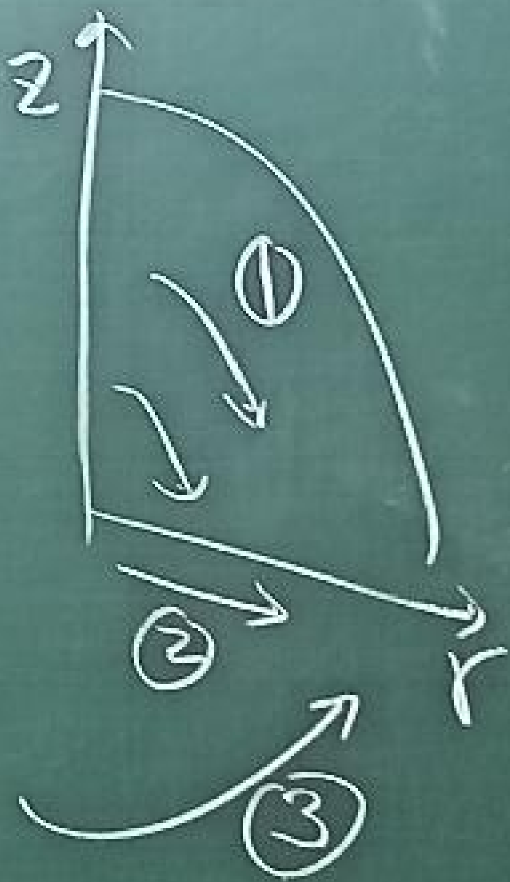
Case I: $d\rho\, d\theta\, d\phi$ or $d\theta\, d\rho\, d\phi$

$\phi = \text{constant}$

Case II $d\theta\, d\phi\, d\rho$ $\quad$ $d\phi\, d\theta\, d\rho$

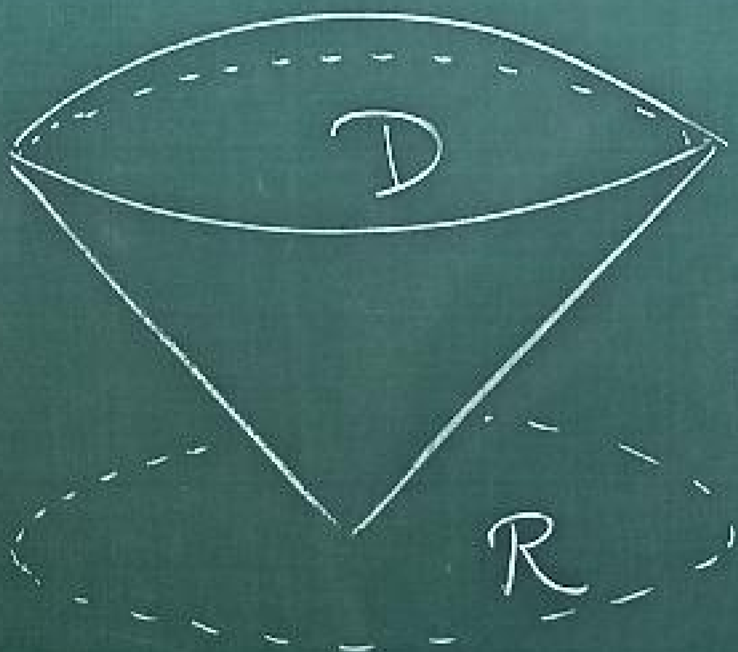
Case III $d\rho\, d\phi\, d\theta$, $d\phi\, d\rho\, d\theta$





Example. $D = \left\{ \begin{array}{l} x^2+y^2+z^2 \le 1 \\ z \ge \sqrt{\dfrac{x^2+y^2}{3}} \end{array} \right\}$

Find volume of D



<u>Sol</u> (I). Cylindrical coordinate

$$R = \left\{ (x,y) \;\exists\, z \in \mathbb{R},\ \sqrt{\frac{x^2+y^2}{3}} \le z \le \sqrt{1-(x^2+y^2)} \right\}$$

$$= \left\{ \frac{x^2+y^2}{3} \le 1-(x^2+y^2) \right\}$$

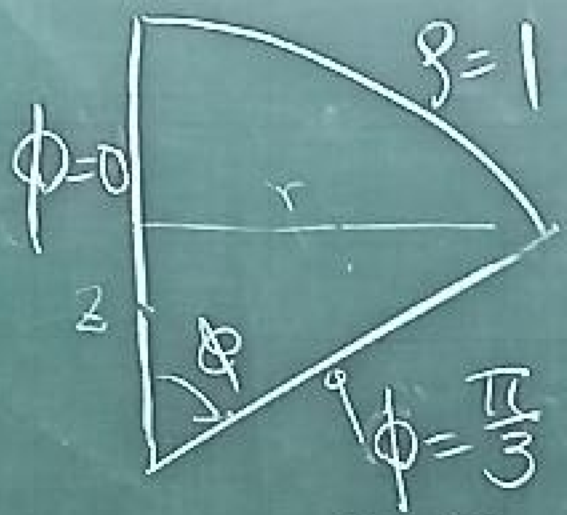
$$= \left\{ x^2+y^2 \le \frac{3}{4} \right\}$$

$$V = \iint_R \int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz\, dA$$

$$= \int_0^{2\pi} \int_{r=0}^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{3}\right) r\, dr\, d\theta$$

II Spherical Coordinate

$(x^2+y^2+z^2=1)$



$\left(\frac{r}{z}=\sqrt{3},\ \tan\phi=\sqrt{3}\right)$

$$z=\sqrt{\frac{x^2+y^2}{3}}$$

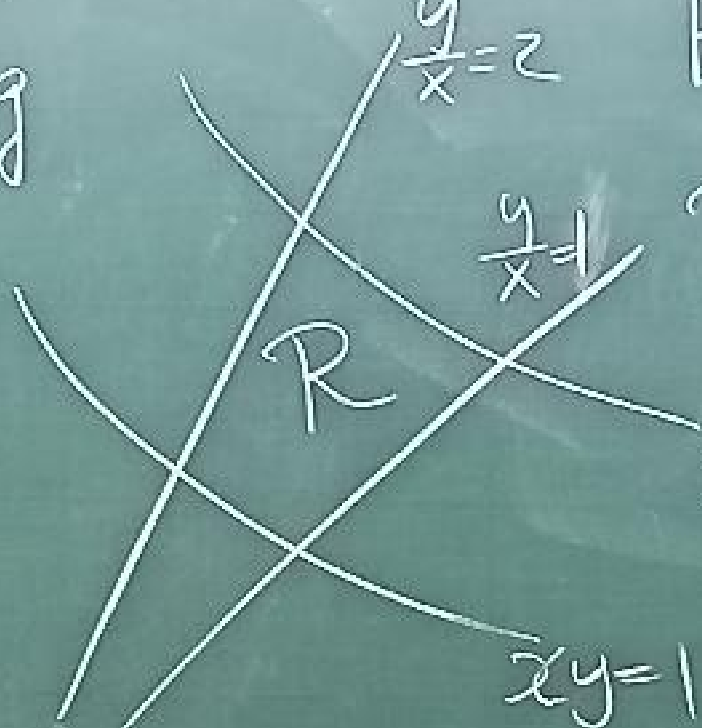
$$D=\left\{\begin{array}{l}0\le\rho\le 1\\ 0\le\phi\le\frac{\pi}{3}\\ 0\le\theta\le 2\pi\end{array}\right\}$$

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\underbrace{\frac{1}{3}}_{d\rho}\underbrace{\left(1-\frac{1}{2}\right)}_{d\phi}\underbrace{2\pi}_{d\theta}=\frac{\pi}{3}$$

Substitution in Multiple Integrals

Eg



Find area of

$$R=\left\{\begin{array}{l}1\le xy\le 2\\ 1\le \frac{y}{x}\le 2\end{array}\right\}=\left\{\begin{array}{l}1\le u\le 2\\ 1\le v\le 2\end{array}\right\}$$

$$A=\int_{v=1}^{2}\int_{u=1}^{2} dA.$$

<u>Q</u>: $dA = \underline{?}\, du\,dv$

<u>Ans</u>: $dA=\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| du\,dv$

$$\underset{J}{\overset{\parallel}{\frac{\partial(x,y)}{\partial(u,v)}}}=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}$$

Need $\left\{\begin{array}{l}u=xy\\ v=\frac{y}{x}\end{array}\right.$

$\longrightarrow$ $x=x(u,v)=\sqrt{\frac{u}{v}}$

$y=y(u,v)=\sqrt{uv}$