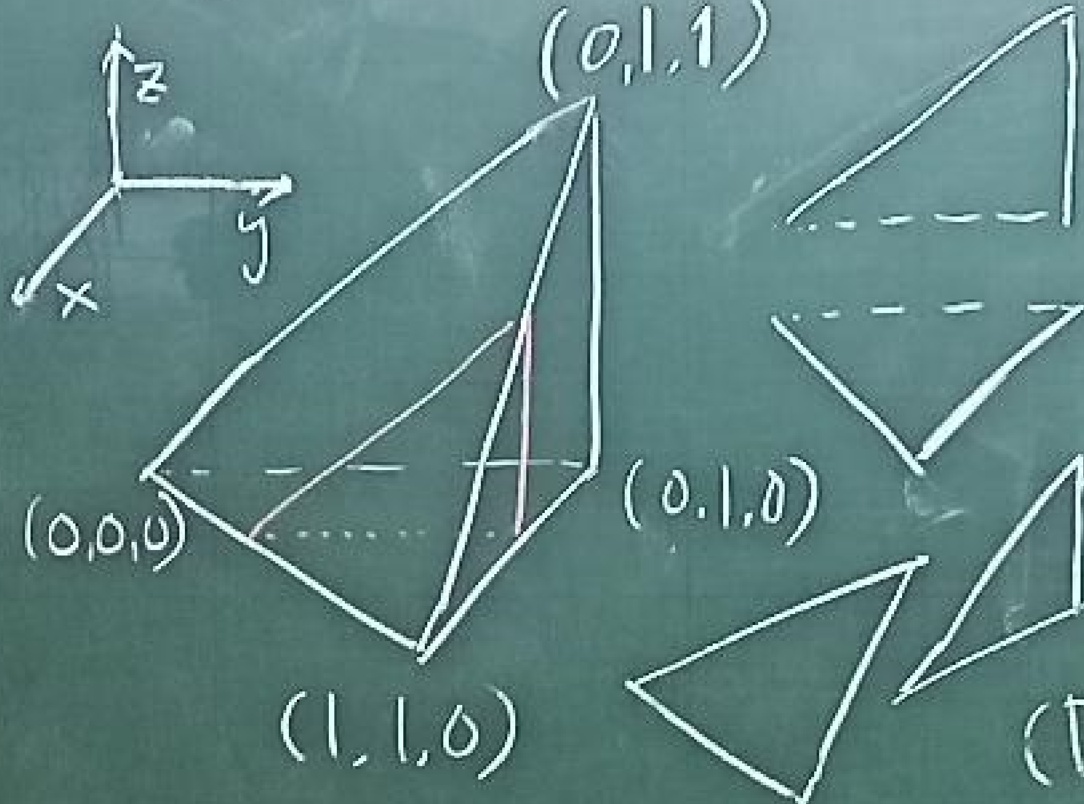
Eg5 continued. Volume = ?



(A): $x=0$

(B): $z=0$

(C) $y=1$

(D) $x-y+z=0$

Case ②: $dz\,dy\,dx$

Step 1 $(z=F_2(x,y))$

$$\int_{(B)}^{(D)} 1\,dz = \int_{z=0\ (F_1(x,y))}^{z=y-x} 1\,dz$$

$$= y-x$$

Step2

$$\int_{B\cap D}^{c}(y-x)\,dy=\int_{y=x\,(y=G_1(x))}^{y=1\,(y=G_2(x))}(y-x)\,dy$$

$$=\int_{y=x}^{1}\left(\frac{y^2}{2}-xy\right)=\frac{1-x^2}{2}-x(1-x)$$

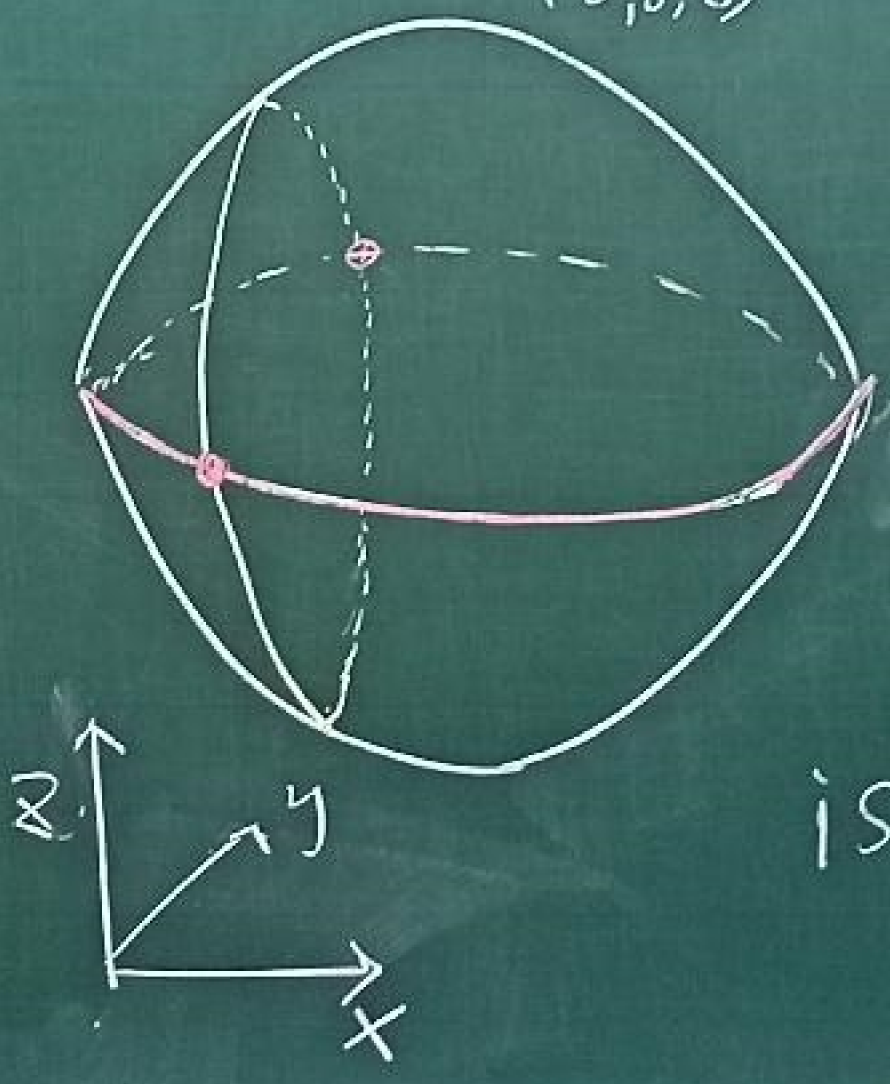
$$=\frac{(1-x)^2}{2}$$

Step3: $\displaystyle\int_0^1\frac{(1-x)^2}{2}\,dx$

$$=-\frac{(1-x)^3}{6}\Big|_0^1=\frac{1}{6}$$

Other cases: homework.

Eg 6: Find the volume of $D = \{x^2+3y^2 \le z \le 8-x^2-y^2\}$



Obviously, either $dz\,dx\,dy$ or $dz\,dy\,dx$ is preferred!

Step 1

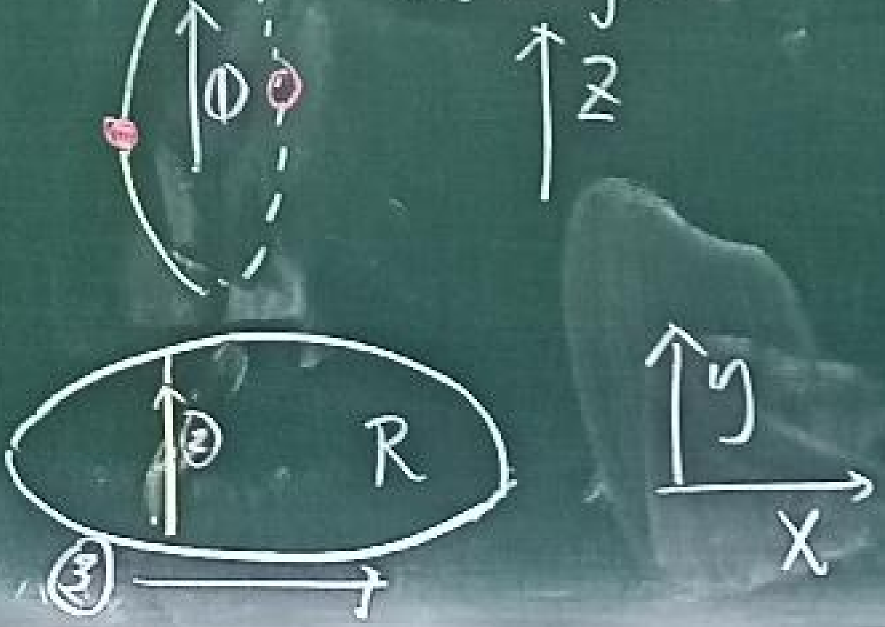
$$V=\iint_R \int_{z=x^2+3y^2}^{8-x^2-y^2} 1\,dz\,dA=\iint_R 8-2x^2-4y^2\,dA$$

$$R=\{x^2+3y^2\le 8-x^2-y^2\}$$

$$\left(\text{So } \int_{x^2+3y^2}^{8-x^2-y^2} dz \text{ is doable}\right)$$

$$=\{x^2+2y^2\le 4\}$$

$$V=\iint_{x^2+2y^2\le 4} 8-2x^2-4y^2\;dx\,dy \text{ or } \underline{dy\,dx}$$



Step2 $\iint_R 8-2x^2-4y^2\,dy\,dx$

$$= \int_{x=-2}^{2}\int_{y=G_1(x)}^{G_2(x)} 8-2x^2-4y^2\,dy\,dx$$

Since
$$R = \left\{ \begin{array}{l} -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}} \\ -2 \le x \le 2 \end{array} \right\}$$

$$= \int_{-2}^{2} (8-2x^2)y - \frac{4y^3}{3} \Bigg|_{y=-\sqrt{\frac{4-x^2}{2}}}^{y=\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} (4-x^2)\cdot 4\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\,\frac{4-x^2}{2}\,2\sqrt{\frac{4-x^2}{2}}\,dx$$

$$= \int_{-2}^{2} \frac{8}{3}(4-x^2)\sqrt{\frac{4-x^2}{2}}\,dx \quad \left(\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta\,d\theta \end{array} \right)$$

Triple Integrals in Cylindrical and Spherical Coordinates

I. Cylindrical Coordinate

= polar coordinate + z coord.

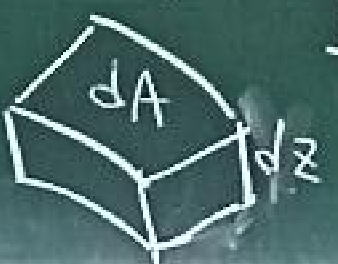
$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$$

$$r = \sqrt{x^2+y^2},\quad \tan\theta = \frac{y}{x},\quad z = z$$

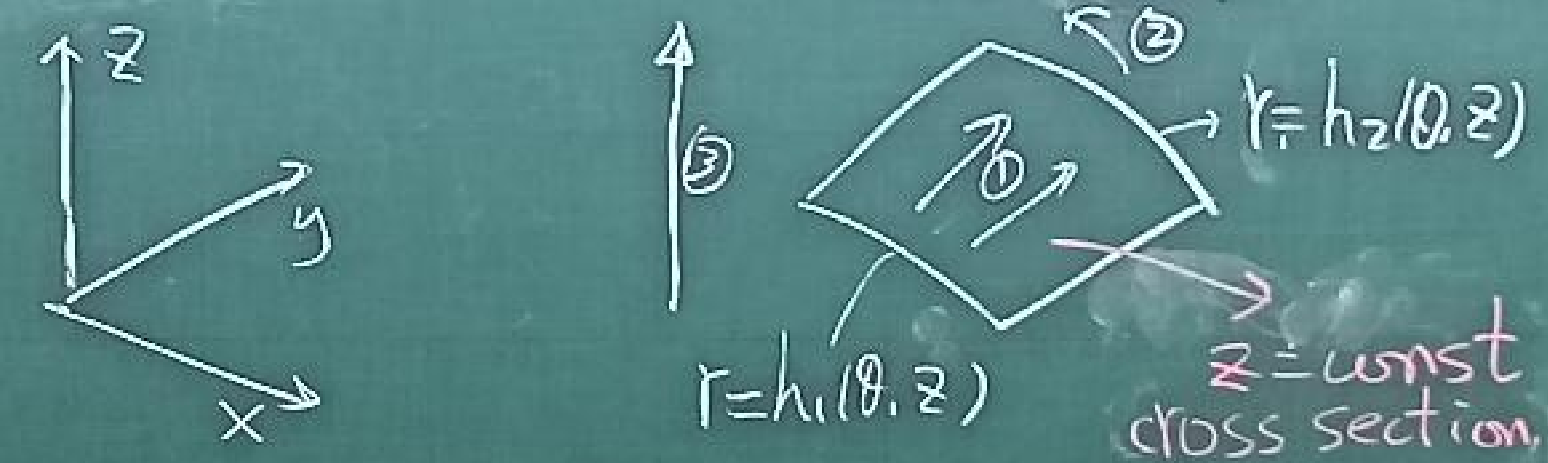
$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), & (x,y) \in \mathrm{I}, \mathrm{IV} \\ \tan^{-1}\frac{y}{x} \pm \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) & (x,y) \in \mathrm{II} \\ \qquad\qquad\quad \left(\frac{-3\pi}{2}, \frac{\pi}{2}\right) & \qquad\quad\ \ \mathrm{III} \end{cases}$$

$$dV = dA \cdot dz = r\,dr\,d\theta\,dz$$

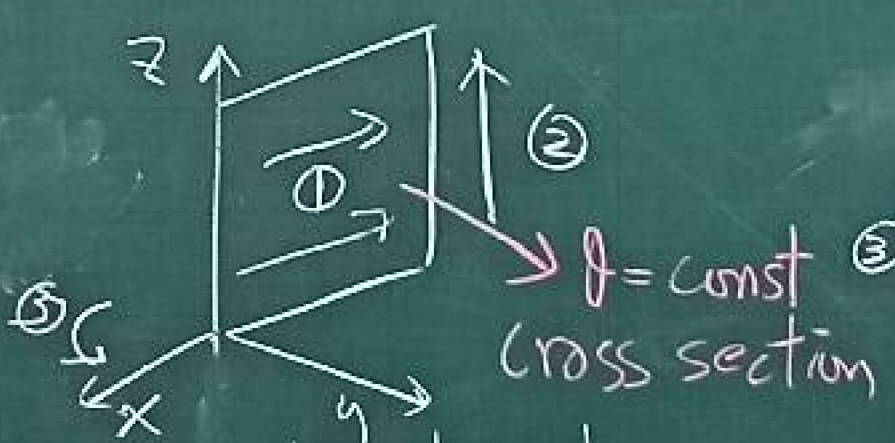


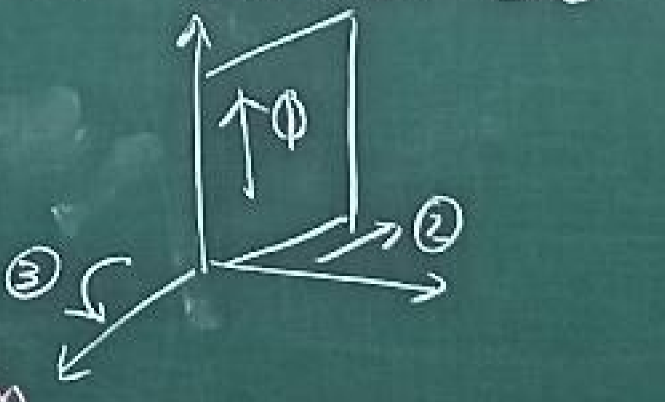
Finding limits of integration

Case I: $r\,dr\,d\theta\,dz$ ($d\theta\,r\,dr\,dz$ is similar)

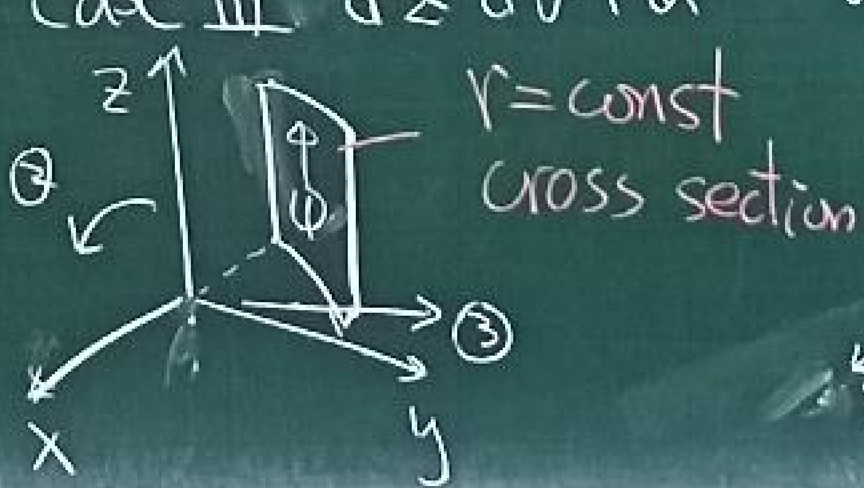


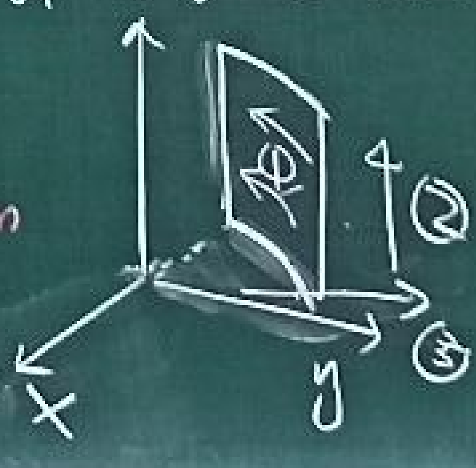
Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$



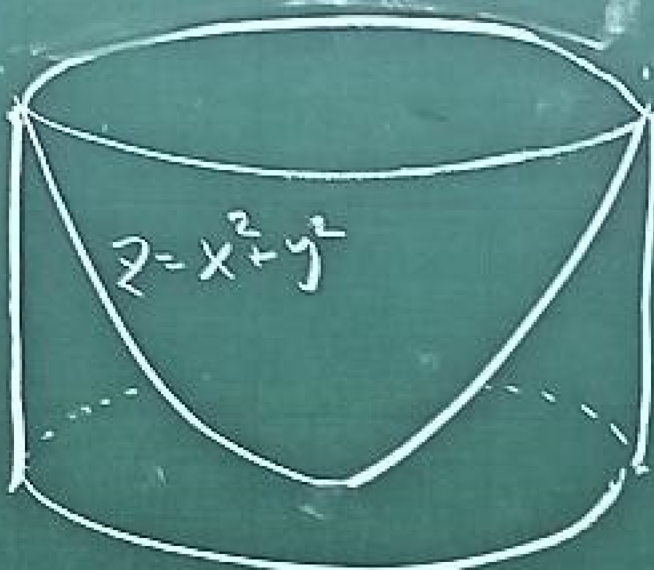


Case III $dz\,d\theta\,r\,dr$ or $d\theta\,dz\,r\,dr$





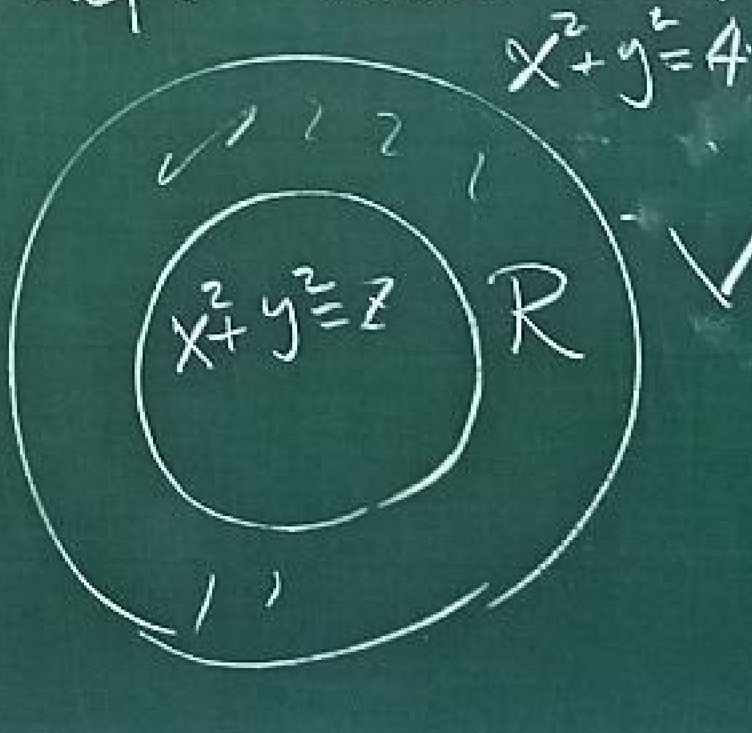
Eg 1: $D = \left\{ \begin{array}{l} x^2+y^2 \le 4 \\ 0 \le z \le x^2+y^2 \end{array} \right\}$



Find volume of D using Cylindrical coordinates.

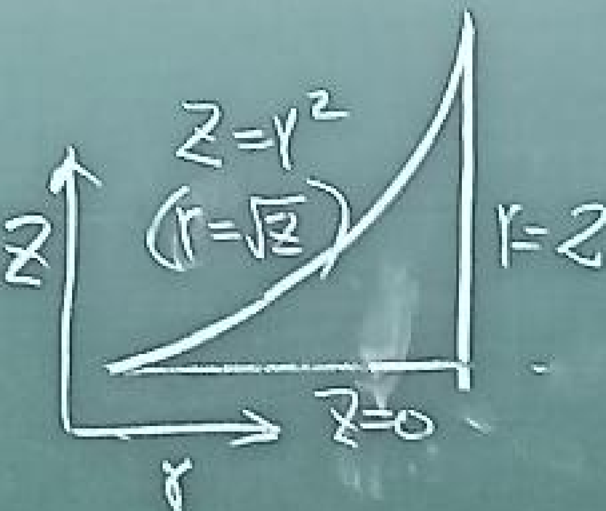
case I rdrdθdz

Step 1 cross section



$$V = \int_0^4 \int_0^{2\pi} \int_{r=\sqrt{z}}^{2} r\,dr\,d\theta\,dz$$

Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$

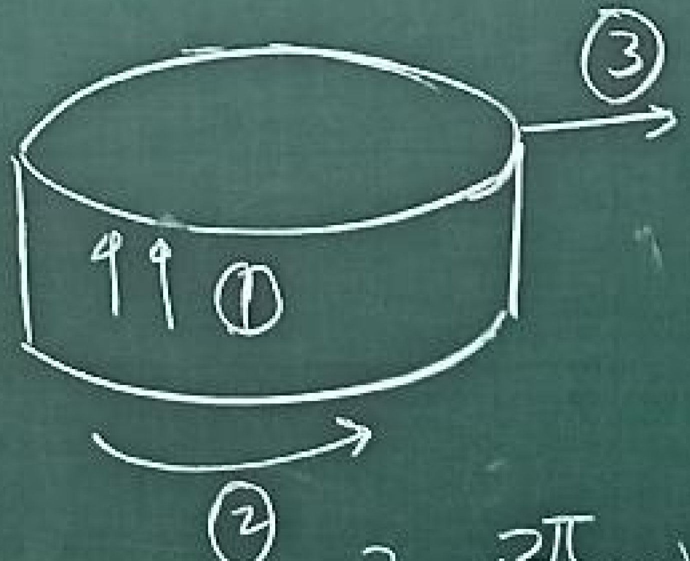


$$V=\int_0^{2\pi}\int_0^{4}\int_{r=\sqrt{z}}^{2} r\,dr\,dz\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2}\int_{z=0}^{r^2} dz\,r\,dr\,d\theta$$

$$=8\pi$$

Case III $dz\,d\theta\,r\,dr$



$$V=\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{r^2} dz\,d\theta\,r\,dr$$

$$=8\pi$$