Integration in Polar Coordinates

Eg 1 $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx$

Sol

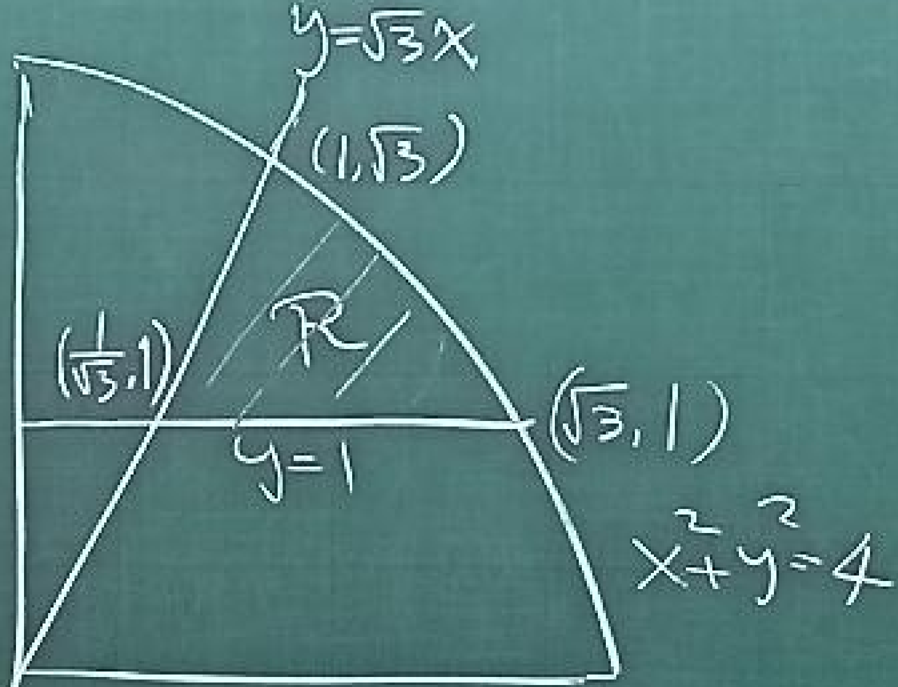
$$R=\left\{\begin{matrix}0\le y\le\sqrt{1-x^2}\\ 0\le x\le 1\end{matrix}\right\}$$

$$=\left\{\begin{matrix}0\le r\le 1\\ 0\le\theta\le\frac{\pi}{2}\end{matrix}\right\}$$

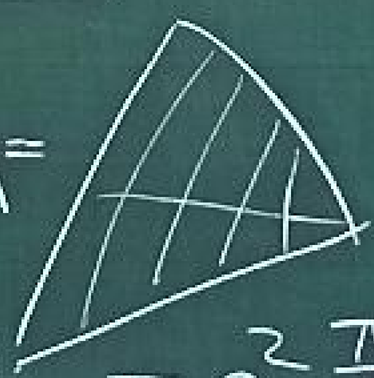
$$I=\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r^2\,(\underbrace{r\,dr\,d\theta}_{=\,dA})$$

$$=\left(\int_0^1 r^3\,dr\right)\left(\int_0^{\frac{\pi}{2}} d\theta\right)=\frac{\pi}{8}$$

Eg2 Find the area enclosed by
$$\begin{cases} y=\sqrt{3}x \\ y=1 \\ x^2+y^2=4 \end{cases}$$
in 1st quadrant

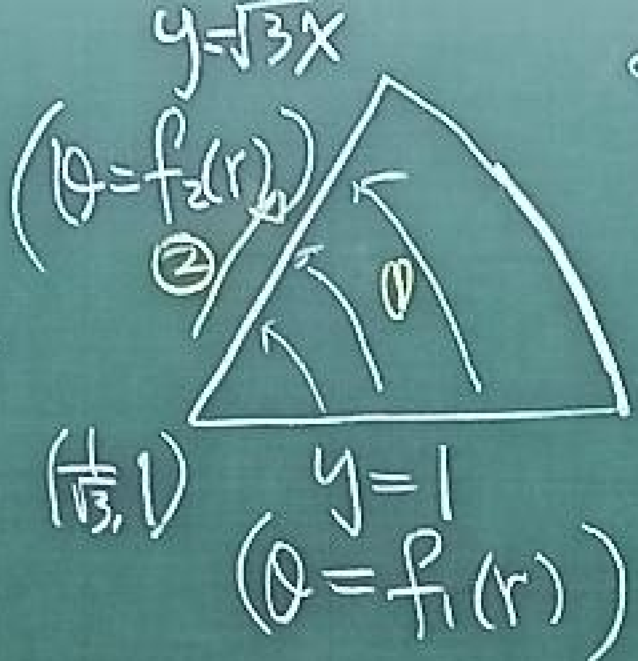


Sol: Method 1:

A =  − 

$$= \pi 2^2 \cdot \frac{\frac{\pi}{6}}{2\pi} - \frac{1}{2}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)\cdot 1$$

Method 2: Polar Coordinate



$$R=\left\{\begin{array}{l} f_1(r)\leq\theta\leq f_2(r) \\ \dfrac{2}{\sqrt{3}}\leq r\leq 2 \end{array}\right\}$$

$$\frac{2}{\sqrt{3}} = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+1^2}$$

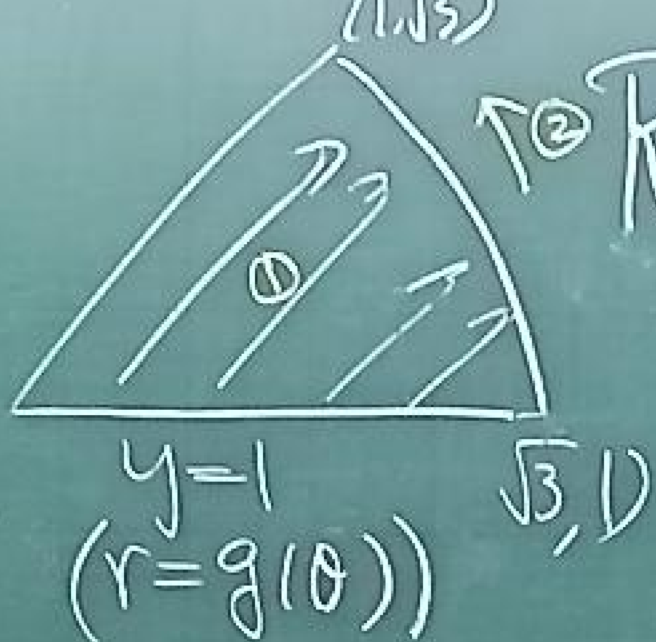
$$y=1 \Leftrightarrow r\sin\theta=1 \Rightarrow \theta=\sin^{-1}\left(\frac{1}{r}\right) = f_1(r)$$

$$y=\sqrt{3}x \Leftrightarrow \theta=\frac{\pi}{3}=f_2(r)$$

$$I=\int_{r=\frac{2}{\sqrt{3}}}^{2}\int_{\theta=\sin^{-1}\left(\frac{1}{r}\right)}^{\frac{\pi}{3}} d\theta\, r\, dr \rightarrow \text{difficult}$$

$$=\int_{\frac{2}{\sqrt{3}}}^{2} r\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{1}{r}\right)\right)dr = ?$$

Method 3



$$R = \left\{ \begin{array}{l} g(\theta) \le r \le 2 \\ \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{3} \end{array} \right\}$$

$$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right), \qquad \frac{\pi}{3} = \tan^{-1}(\sqrt{3})$$

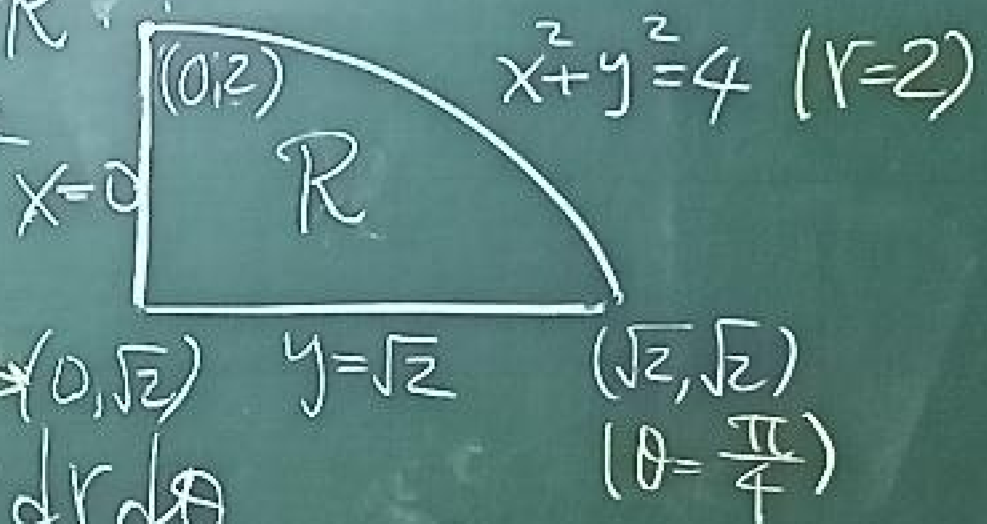
$$y = 1 \Leftrightarrow r\sin\theta = 1 \Rightarrow r = \frac{1}{\sin\theta} = g(\theta)$$

$$I = \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{r=\frac{1}{\sin\theta}}^{2} r\,dr\,d\theta$$

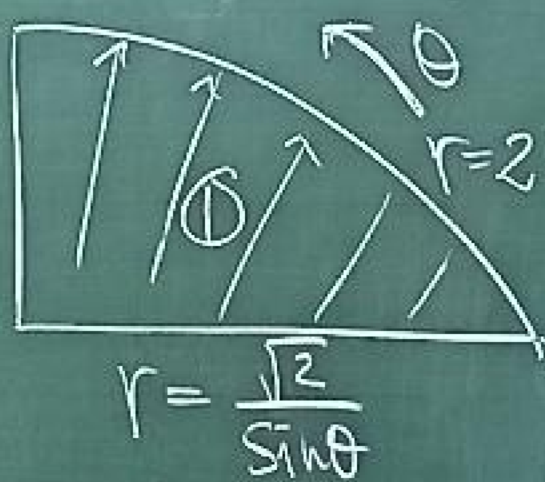
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(2 - \frac{1}{2\sin^2\theta}\right) d\theta$$

$$= \left(2\theta + \frac{\cot\theta}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{1}{2}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)$$

Eg 3: Evaluate $\iint_R f(x,y)\,dA$ in polar coordinate where

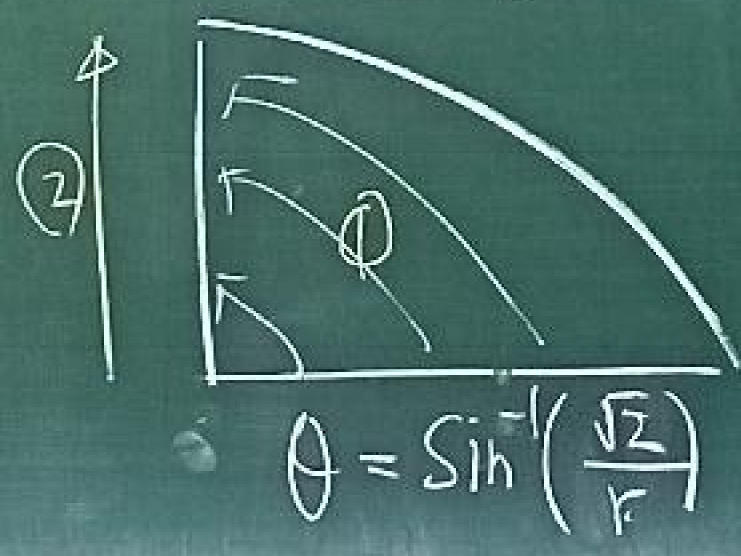


Sol: Method ⓐ $r\,dr\,d\theta$



$$I=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=\frac{2}{\sin\theta}}^{2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

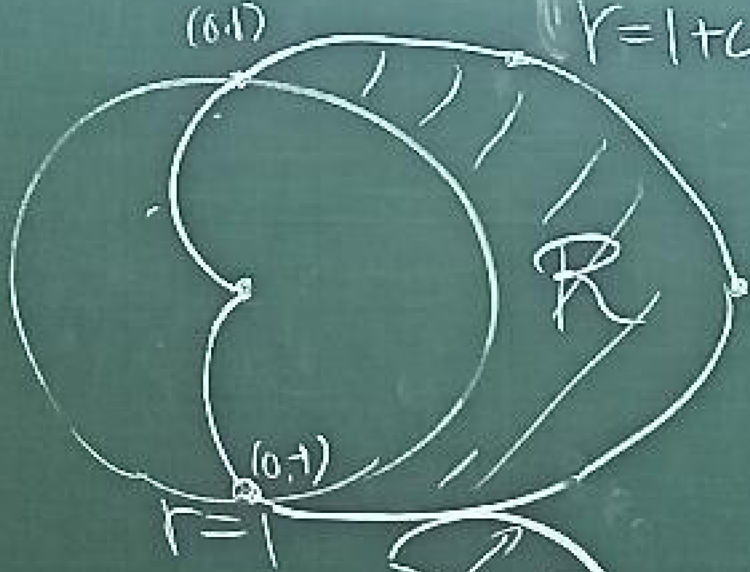
Method ⓑ $d\theta\,r\,dr$



$$I=\int_{r=\sqrt{2}}^{2}\int_{\theta=\sin^{-1}\left(\frac{\sqrt{2}}{r}\right)}^{\frac{\pi}{2}} f(r\cos\theta, r\sin\theta)\,d\theta\,r\,dr$$

Eg 4 Evaluate $\iint_R g(r,\theta)\,dA$

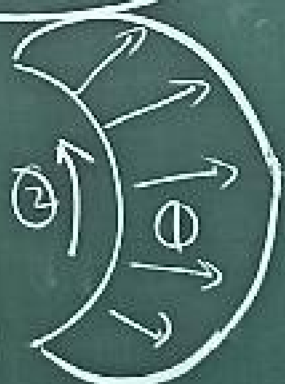
where R is the region $\begin{cases} \text{inside } r = 1+\cos\theta \\ \text{outside } r = 1 \end{cases}$



$r = 1+\cos\theta \Rightarrow \cos\theta = r-1$

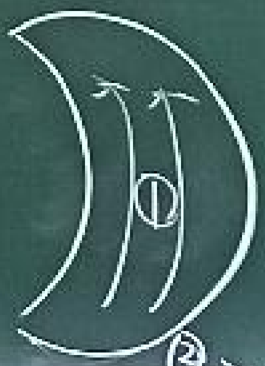
$\Rightarrow \theta = \begin{cases} \cos^{-1}(r-1) & \text{if } \theta \in [0, \frac{\pi}{2}] \\ -\cos^{-1}(r-1) & \text{if } \theta \in [\frac{-\pi}{2}, 0] \end{cases}$

(a) $r\,dr\,d\theta$: (usually preferred)



$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} g(r,\theta)\, r\,dr\,d\theta$$

(b) $d\theta\, r\,dr$



$$I = \int_{r=1}^{2} \int_{-\cos^{-1}(r-1)}^{\cos^{-1}(r-1)} g(r,\theta)\, d\theta\, r\,dr$$

Remark

$r\,dr\,d\theta$ is usually a better choice than $d\theta\, r\,dr$ since most curves in polar coordinates appear naturally in the form $r = f(\theta)$, which is needed in $\int_{(*)}^{(*)} r\,dr$

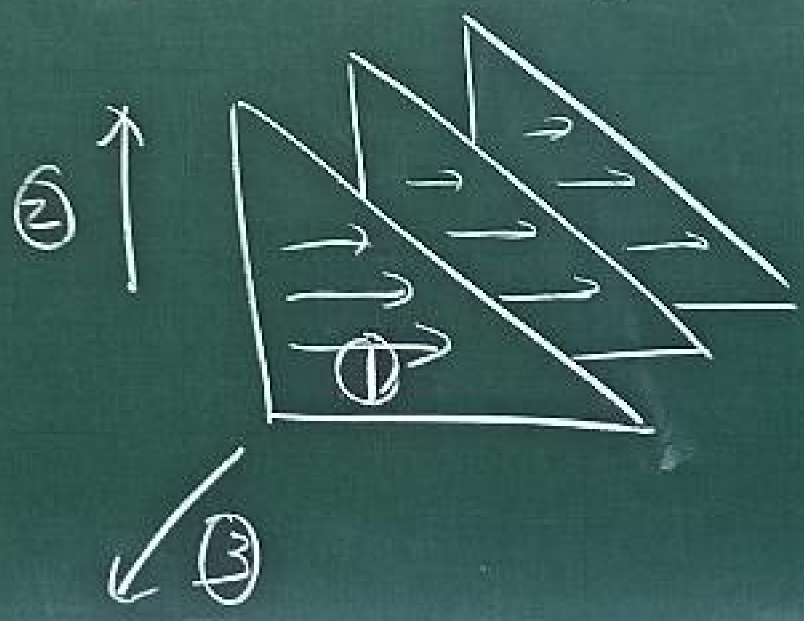
Triple integrals in Cartesian coordinates

$$I = \iiint_D f(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_{k=0}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

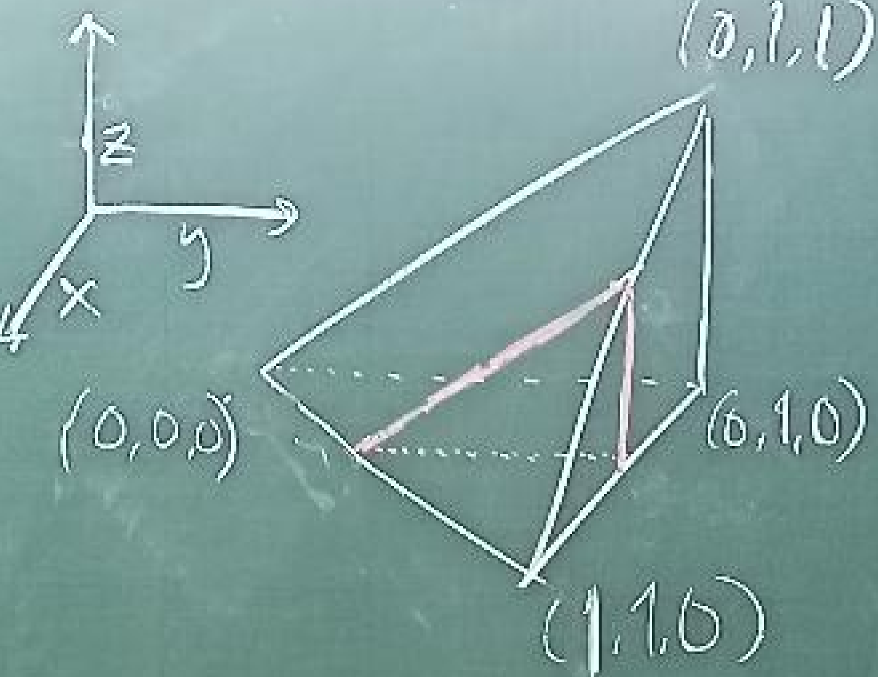
when $f(x,y,z) = 1$, $I =$ volume of D

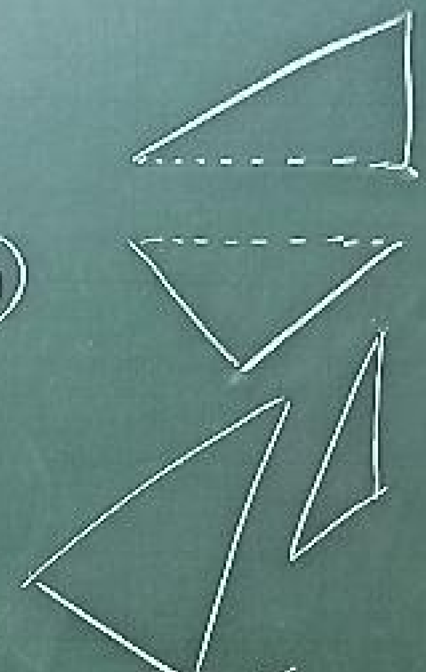
In Cartesian coordinates

$$dV = dx\,dy\,dz = dy\,dz\,dx = \cdots$$



Eg 5. Find the volume of the domain enclosed by





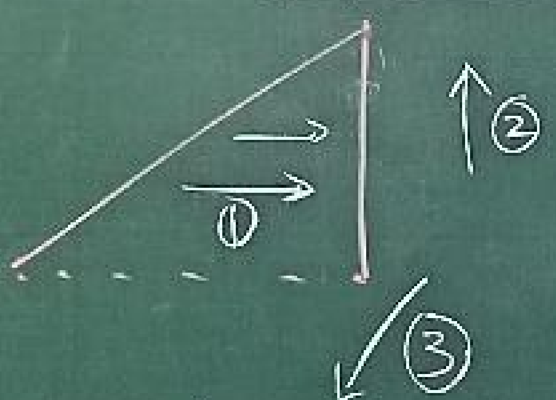
$x = 0 \cdots (A)$

$z = 0 \cdots (B)$

$y = 1 \cdots (C)$

$x - y + z = 0 \cdots (D)$

Case ① $\underline{dy\,dz\,dx}$



$$I = \int_{x=0}^{1} \int_{z=0}^{z=1-x} \int_{y=x+z}^{y=1} 1 \, dy\, dz\, dx$$

dy: need C, D in the form: $y = f(x,z)$

D: $y = x + z$, C: $y = 1$

dz: B, $C \cap D$: $z = g(x)$, B: $z = 0$, $C \cap D$: $z = 1 - x$