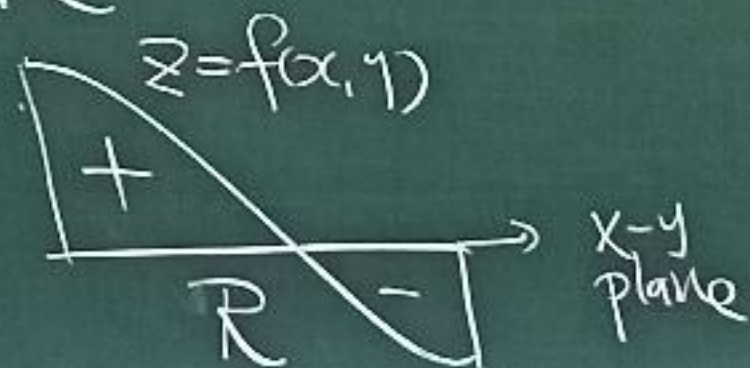
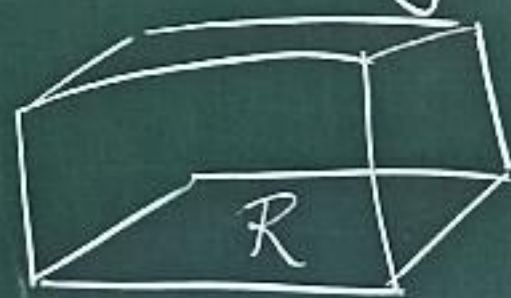


Double Integral

Def: Let $R = [a, b] \times [c, d]$
 $= \{ a \leq x \leq b, c \leq y \leq d \}$

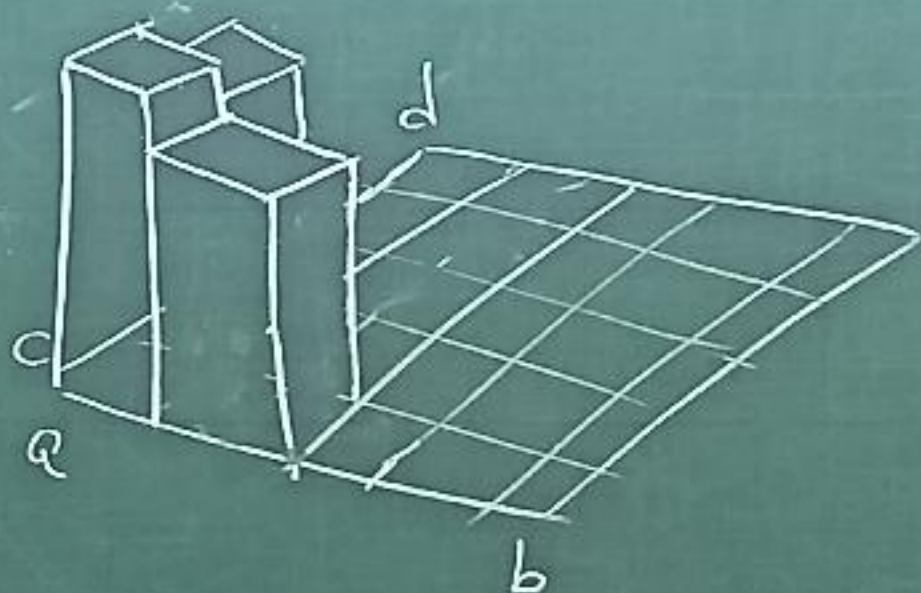
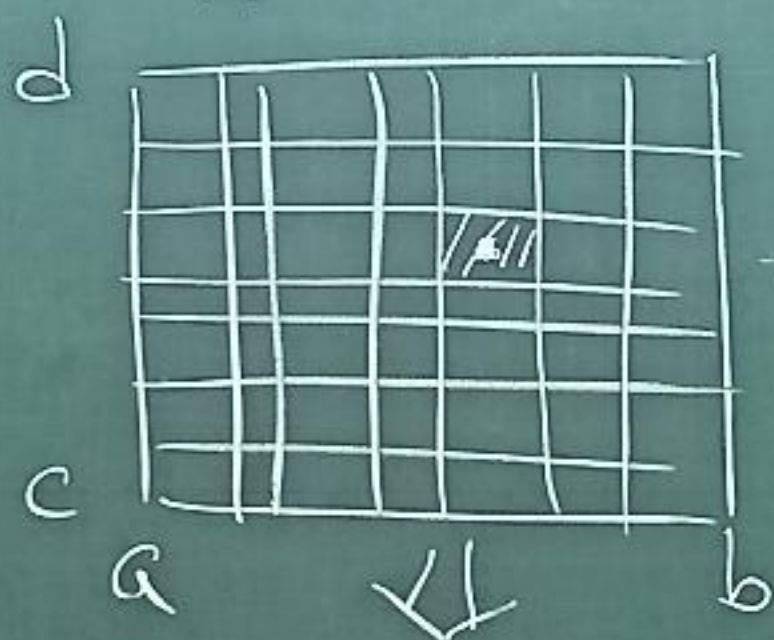
$$\iint_R f(x, y) dA$$


= Signed Volume between
 $z = f(x, y)$ graph and x - y plane
over the region R .



In other words,

$$\iint_R f(x, y) dA = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$



R_k :  - kth rectangle
Area $\triangleq \Delta A_k$

$$P = \left\{ \begin{array}{l} a = x_0 < x_1 < \dots < x_n = b \\ c = y_0 < y_1 < \dots < y_n = d \end{array} \right\}$$

$$\|P\| = \max \left\{ \underbrace{\Delta x_i}_{x_i - x_{i-1}}, \underbrace{\Delta y_j}_{y_j - y_{j-1}} \right\}$$

$$(x_k, y_k) \in R_k$$

In general, if $f(x, y)$ is continuous on R , then

$\iint_R f(x, y) dA$ exists.

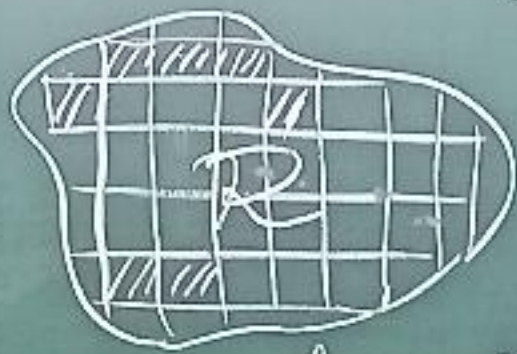
(f is integrable on R)

Fubini's Theorem.

If f is continuous on $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

General Region R

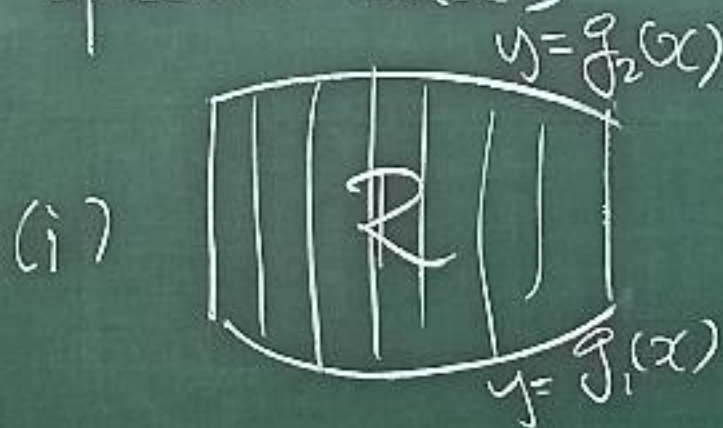


$$\iint_R f(x, y) dA$$

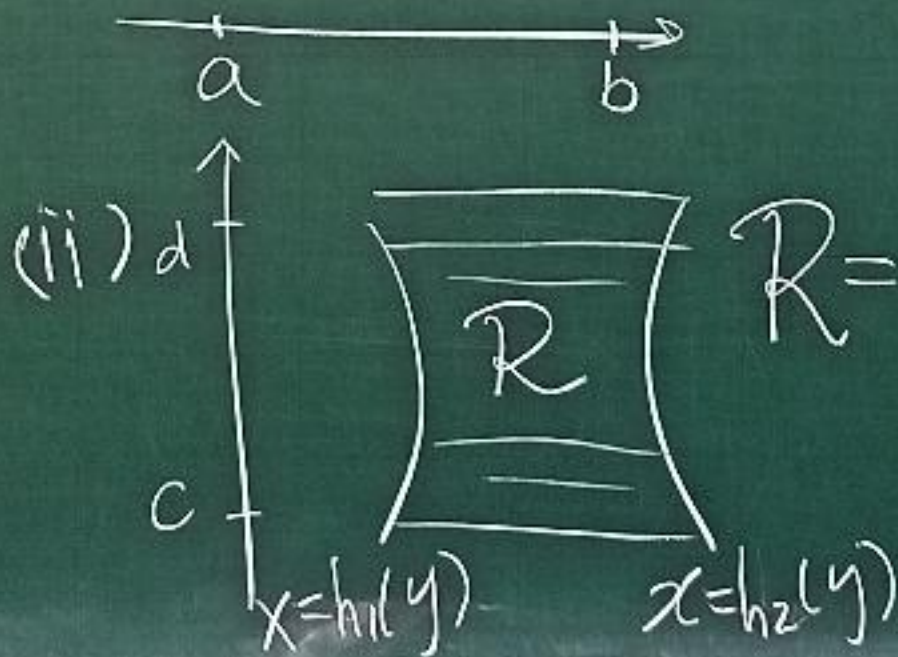
$$= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

$R_k \subset R$

Special cases



$$R = \left\{ \begin{array}{l} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{array} \right\}$$



$$R = \left\{ \begin{array}{l} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{array} \right\}$$

Fubini's Theorem (Strong form)

If f is cont. on R and

$$(i) R = \{a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

$$\text{Then } \iint_A f(x,y) dA = \int_{x=a}^b \int_{y=g_1(x)}^{g_2(x)} f(x,y) dy dx$$

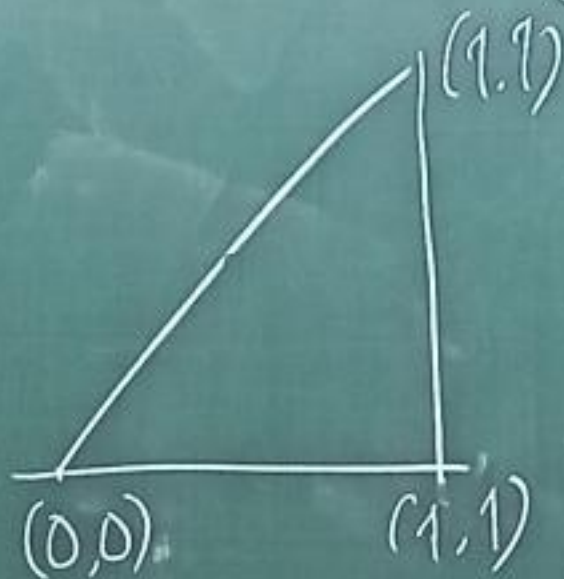
or

$$(ii) R = \{c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

$$\text{Then } \iint_A f(x,y) dA = \int_{y=c}^d \int_{x=h_1(y)}^{h_2(y)} f(x,y) dx dy$$

Ex 1 R = region enclosed by

$$x=y, x=1 \text{ and } y=0$$



$$\iint_R \frac{\sin x}{x} dA = ?$$

Sol: $R = \{0 \leq x \leq 1, 0 \leq y \leq x\}$  ⁽ⁱ⁾

$$= \{0 \leq y \leq 1, y \leq x \leq 1\}$$
  _(ii)

$$(ii) = \int_{y=0}^1 \int_{x=y}^1 \frac{\sin x}{x} dx dy = ?$$

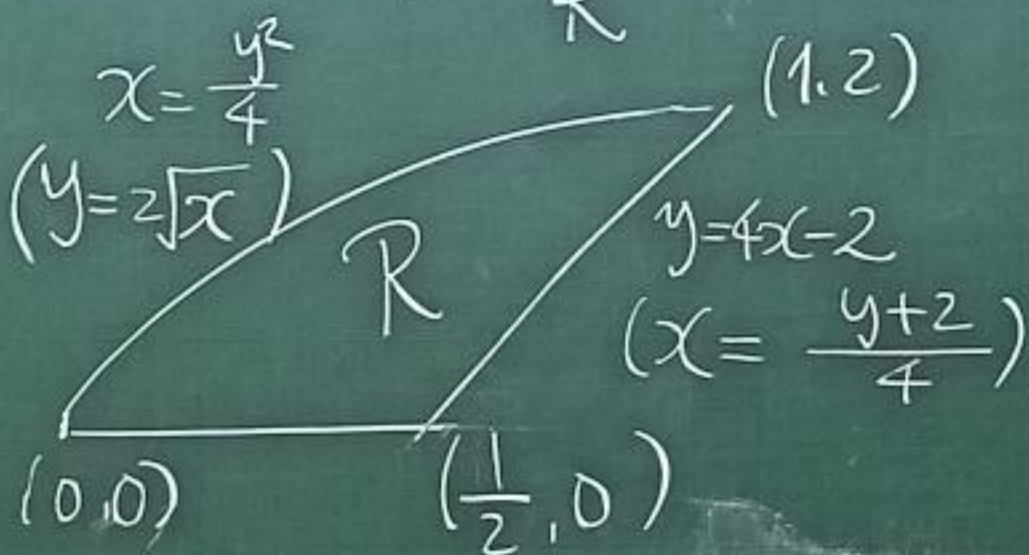
$$(i) = \int_{x=0}^1 \int_{y=0}^x \frac{\sin x}{x} dy dx$$

$$= \int_0^1 \left(\frac{\sin x}{x} \int_{y=0}^x 1 dy \right) dx = -\cos x \Big|_0^1 = 1 - \cos 1$$

Ex 2: Find the area of

the region enclosed by $\begin{cases} y=0 \\ y=4x-2 \\ x=\frac{y^2}{4} \end{cases}$
($y > 0$)

Sol: $A = \iint_R 1 \, dA$



(I): exercise.

$$(II) = \int_{x=0}^1 \int_{y=0}^{2\sqrt{x}} 1 \, dy \, dx - \frac{1}{2} \cdot \frac{1}{2} \cdot 2$$

$$= \int_0^1 2\sqrt{x} \, dx - \frac{1}{2} = \frac{5}{6}$$

$$(III) = \int_{y=0}^2 \int_{x=\frac{y^2}{4}}^{\frac{y+2}{4}} 1 \, dx \, dy$$

$$= \int_{y=0}^2 \left(\frac{y+2}{4} - \frac{y^2}{4} \right) dy$$

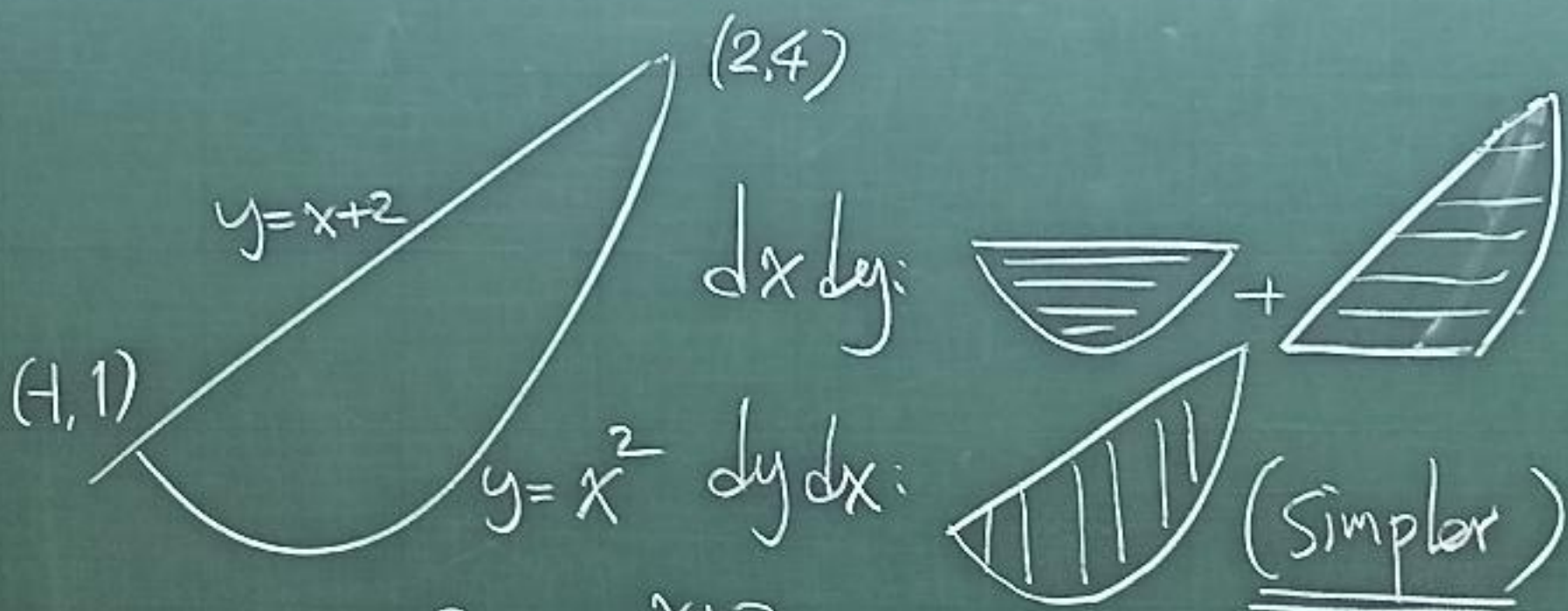
$$= \int_0^2 \left(-\frac{y^2}{4} + \frac{y}{4} + \frac{1}{2} \right) dy$$

$$= \left. -\frac{y^3}{12} + \frac{y^2}{8} + \frac{y}{2} \right|_{y=0}^2$$

$$= \frac{5}{6}$$

Eg 3. Find area of the region

enclosed by $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

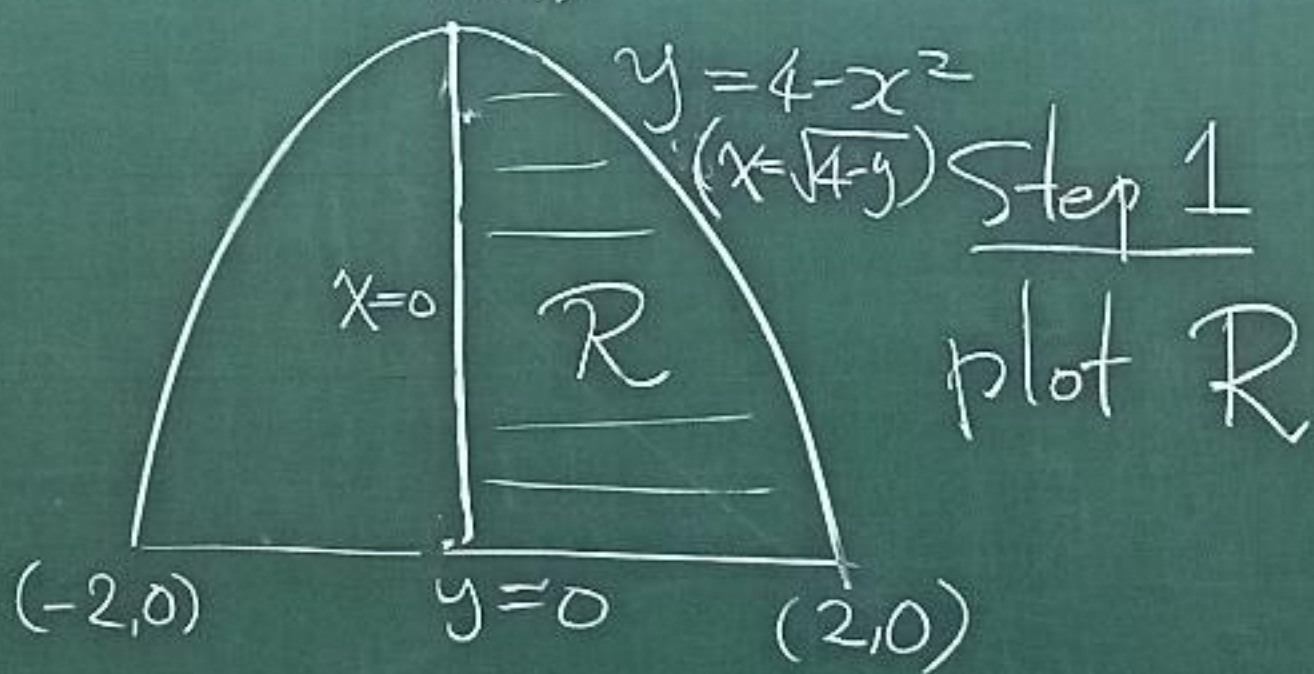


$$A = \int_{x=-1}^2 \int_{y=x^2}^{x+2} dy dx = \int_{-1}^2 (x+2-x^2) dx$$
$$= \left. \frac{x^2}{2} + 2x - \frac{x^3}{3} \right|_{-1}^2 = \frac{9}{2}$$

$$\text{Eg 4: } \int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} dy dx = ?$$

Sol: Try $dx dy$ instead.

Note: $\neq \int_0^{4-x^2} \int_0^2 \dots dx dy$



Step 2: Find limits of integration for dx

left: $x = 0$

Right: $y = 4 - x^2$, $x = \sqrt{4 - y}$

Step 3

$$A = \int_{y=0}^4 \int_{x=0}^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} dx dy$$

$$= \int_{y=0}^4 \frac{e^{2y}}{4-y} \int_{x=0}^{\sqrt{4-y}} x dx dy$$

$$= \int_{y=0}^4 \frac{e^{2y}}{4-y} \cdot \left(\frac{x^2}{2} \Big|_0^{\sqrt{4-y}} \right) dy$$

$$= \frac{1}{2} \int_0^4 e^{2y} dy$$

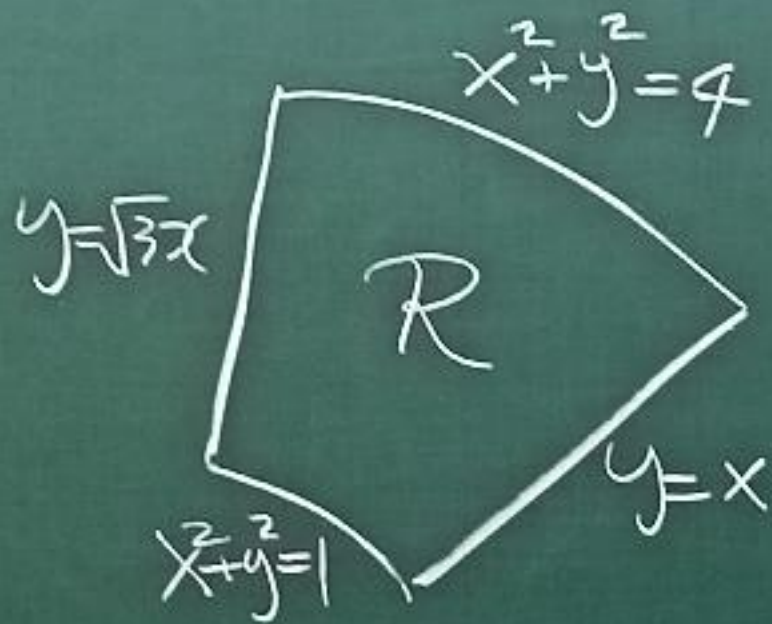
$$= \frac{1}{4} e^{2y} \Big|_0^4 = \frac{e^8 - 1}{4}$$

Integration in polar coordinate.

Eg 5 $\iint_R f(x, y) dA$

R = region enclosed by

$$\begin{cases} y = x \\ y = \sqrt{3}x \\ x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \end{cases}$$



Both  $(dx dy)$

and  $(dy dx)$

are complicated.

In this example

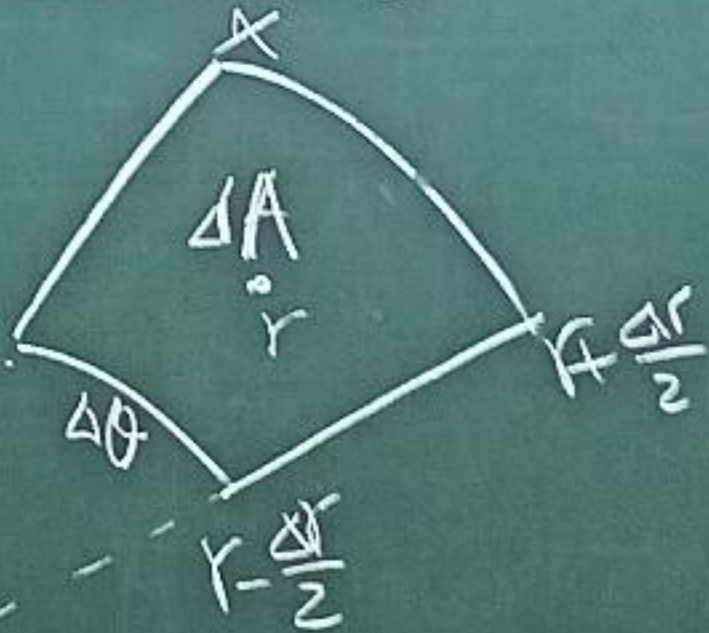
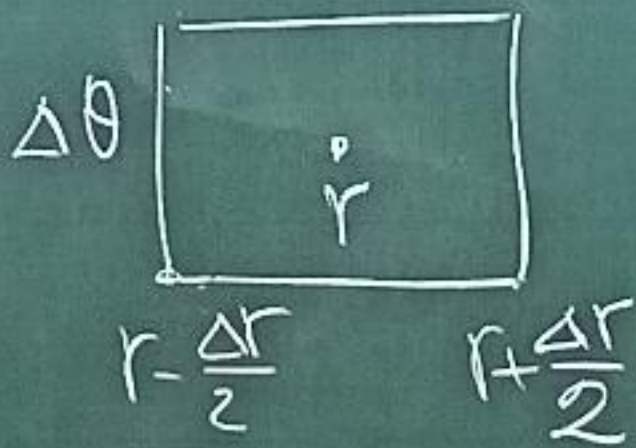
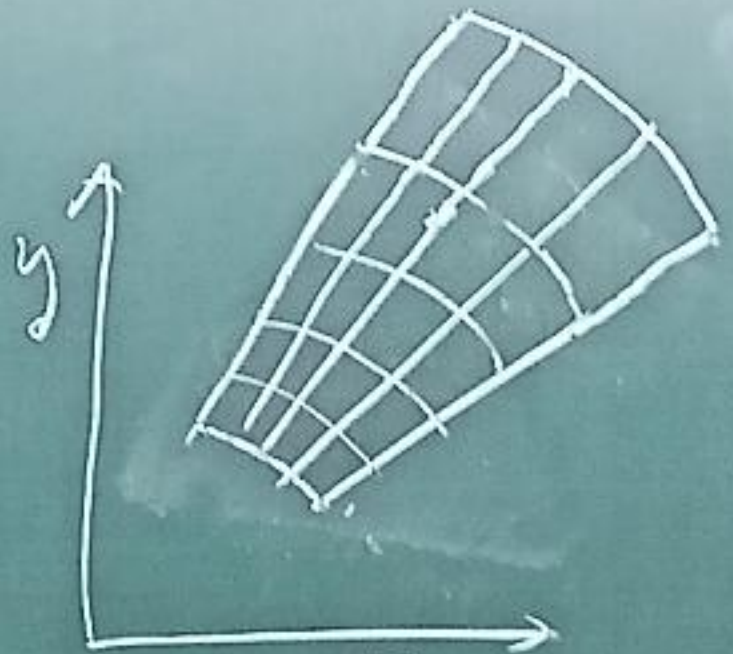
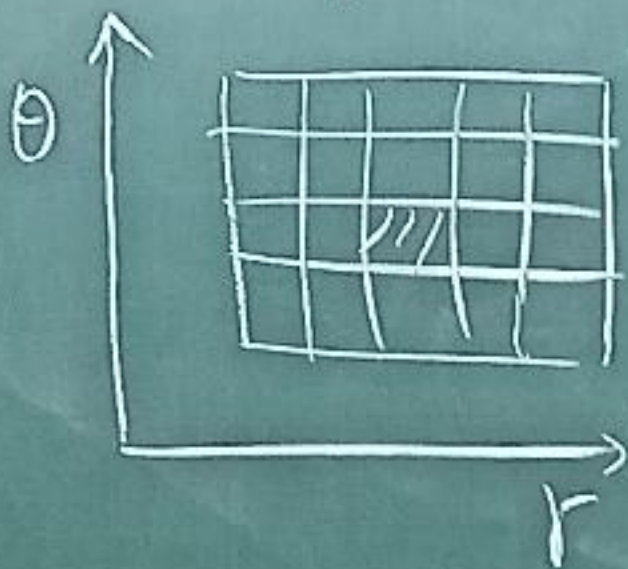
$$R = \left\{ 1 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3} \right\}$$

$$I = \int_{\theta = \frac{\pi}{4}}^{\frac{\pi}{3}} f(\underbrace{r \cos \theta}_x, \underbrace{r \sin \theta}_y) dA$$

$$dA \neq dr d\theta$$

Ans: $dA = r dr d\theta$

Why?



$$\begin{aligned}\Delta A &= \pi \left(r + \frac{\Delta r}{2}\right)^2 \frac{\Delta \theta}{2\pi} - \pi \left(r - \frac{\Delta r}{2}\right)^2 \frac{\Delta \theta}{2\pi} \\ &= r \Delta r \Delta \theta\end{aligned}$$