Method of Lagrangian Multiplier

Eg 1 Find nearest point on the curve $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$ to the origin.

Sol: Minimize $f(x,y,z)=x^2+y^2+z^2$

subject to $\begin{cases} g_1 = x+y+z=1 \\ g_2 = x^2+y^2=1 \end{cases}$

$$x+y+z=1 \quad \text{—— ①}$$
$$x^2+y^2=1 \quad \text{—— ②}$$
$$2x=\lambda_1 1+\lambda_2 2x \quad \text{—— ③}$$
$$2y=\lambda_1 1+\lambda_2 2y \quad \text{—— ④}$$
$$2z=\lambda_1 1 \quad \text{—— ⑤}$$

From ⑤ → ③ ④

$x(1-\lambda_2)=z$

$y(1-\lambda_2)=z$

$\Rightarrow$ ⓐ $\lambda_2=1,\quad z=0$

or ⓑ $\lambda_2\neq 1.\quad x=y$

Case ⓐ

① ② $\Rightarrow \begin{cases} x+y=1 \\ x^2+y^2=1 \end{cases}$

$\Rightarrow (x,y,z)=\begin{cases}(1,0,0)\\(0,1,0)\end{cases}$

Case ⓑ $(x=y)$

② $\Rightarrow x=y=\pm\frac{1}{\sqrt{2}}$

Ans

$z=1-x-y=1\mp\sqrt{2}$

$(1,0,0),\ (0,1,0),\ \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1-\sqrt{2}\right),\ \left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}},1+\sqrt{2}\right)$

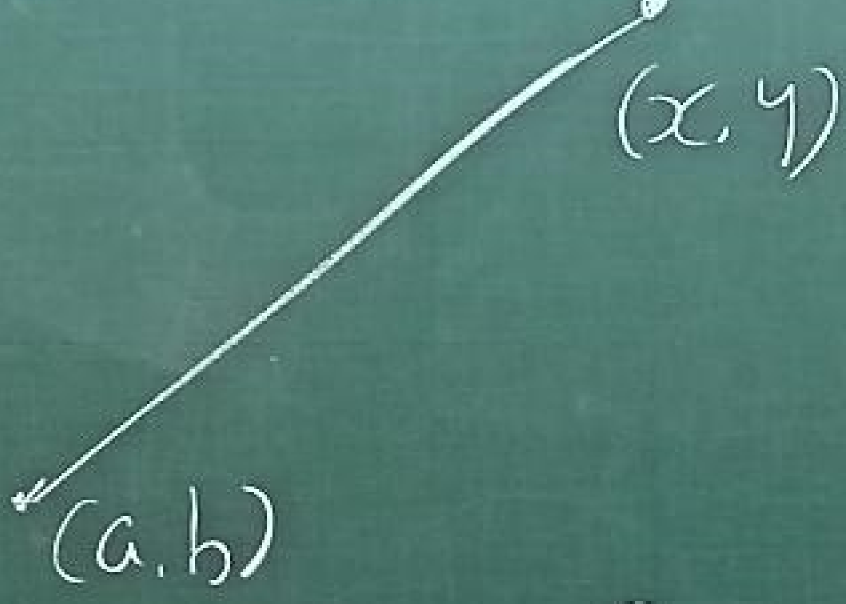
$f=1,\quad 1,\quad 4-2\sqrt{2},\quad 4+2\sqrt{2}$

abs. min　　local max　　abs. max

nearest points

Taylor's Polynomial of $f(x, y)$ centered at (a, b)

$x = a + h, \quad y = b + k$



Let $F(t) = f(a + th, b + tk)$

$h = x - a, \quad k = y - b, \quad t \in \mathbb{R}$

$F(0) = f(a, b), \quad F(1) = f(x, y)$

(Assuming f and all partial derivatives of f are continuous)

Taylor's Thm for $F(t)$

$$F(t) = P_n(t) + R_n(t)$$

where $P_n(t) = \sum_{k=0}^{n} \frac{F^{(k)}(0)}{k!} t^k$

$$R_n(t) = \frac{F^{(n+1)}(c)}{(n+1)!} t^{n+1}$$

where c is between 0 and t

Take $t = 1$.

$$P_n(1) = \sum_{k=0}^{n} \frac{F^{(k)}(0)}{k!}$$

$$R_n(1) = \frac{F^{(n+1)}(c)}{(n+1)!}$$

where $0 < c < 1$

$$F^{(k)}(0) = ?$$

$$F'(t) = f_x(a+th, b+tk)\frac{dx}{dt} + f_y(a+th, b+tk)\frac{dy}{dt}$$

$$x = a+th, \quad y = b+tk$$

$$= (h\partial_x f + k\partial_y f)(a+th, b+tk)$$

$$= (h\partial_x + k\partial_y) f(a+th, b+tk)$$

$$\text{"}\ \frac{d}{dt}F = (h\partial_x + k\partial_y) f\ \text{"}$$

$$F''(t) = h(h\partial_x + k\partial_y)\partial_x f + k(h\partial_x + k\partial_y)\partial_y f$$

$$= (h\partial_x + k\partial_y)\big(\underbrace{(h\partial_x + k\partial_y) f(a+th, b+tk)}_{F'(t)}\big)$$

$$= (h^2\partial_x^2 + 2hk\,\partial_x\partial_y + k^2\partial_y^2) f$$

In summary

$$F''(t) = (h\partial_x + k\partial_y)^2 f(a+th, b+tk)$$

$$F'''(t) = (h\partial_x + k\partial_y)\Big(\underbrace{(h\partial_x + k\partial_y)^2 f(a+th, b+tk)}_{F''(t)}\Big)$$

$$= \underbrace{(h\partial_x + k\partial_y)^3}_{h^3\partial_x^3 + 3h^2k\partial_x^2\partial_y + 3hk^2\partial_x\partial_y^2 + k^3\partial_y^3} f(a+th, b+tk)$$

$$F^{(k)}(t) = (h\partial_x + k\partial_y)^k f(a+th, b+tk)$$

$$\therefore f(x,y) = \tilde{P}_n(x,y) + \tilde{R}_n(x,y)$$

here $\tilde{P}_n(x,y) = P_n(1) = \sum_{k=0}^{n} \frac{(h\partial_x + k\partial_y)^k}{k!} f(a,b)$

$$\tilde{R}_n(x,y) = R_n(1) = \frac{(h\partial_x + k\partial_y)^{n+1}}{(n+1)!} f(a+ch, b+ck), \quad 0<c<1$$

Similarly for $g(x, y, z)$

Let $G(t)$

$= g(x_0 + t\Delta x, y_0 + t\Delta y, z_0 + t\Delta z)$

$\Delta x = x - x_0, \Delta y = y - y_0, \Delta z = z - z_0$

$$\left(\frac{d}{dt}\right)^k G = (\Delta x \partial_x + \Delta y \partial_y + \Delta z \partial_z)^k g$$

Eg 1: Error of linearization for $f(x, y)$.

Error $= f(x, y) - L(x, y)$

$L(x, y) = \tilde{P}_1(x, y)$

$|\text{Error}| = |\tilde{R}_1(x, y)|$

$\leq \frac{1}{2}(|f_{xx}| \Delta x^2 + 2|f_{xy}| |\Delta x \Delta y| + |f_{yy}| \Delta y^2)$

Eg2: Local min/max of f

$$\Delta f = \Delta_1 + \Delta_2 + \Delta_3$$

or

$$f(x,y) = \underbrace{\underset{\substack{\shortparallel \\ k=0}}{f(x_0,y_0)} + \underset{\substack{\shortparallel \\ k=1}}{\Delta_1} + \underset{k=2}{\Delta_2}} + \Delta_3$$

in $\tilde{P}_2(x,y)$ on page 6

Here $|\Delta_3| = |\tilde{R}_2(x,y)|$

$$\leq \frac{1}{3!}\left(|\partial_x^3 f||\Delta x|^3 + 3|\partial_x^2 \partial_y f||\Delta x^2 \Delta y| + 3|\partial_x \partial_y^2 f||\Delta x \Delta y^2| + |\partial_y^3 f||\Delta y|^3\right)$$

$$\leq \frac{M}{3!}(|\Delta x| + |\Delta y|)^3$$

if $|\partial_x^3 f|, |\partial_x^2 \partial_y f|, |\partial_x \partial_y^2 f|, |\partial_y^3 f| \leq M$