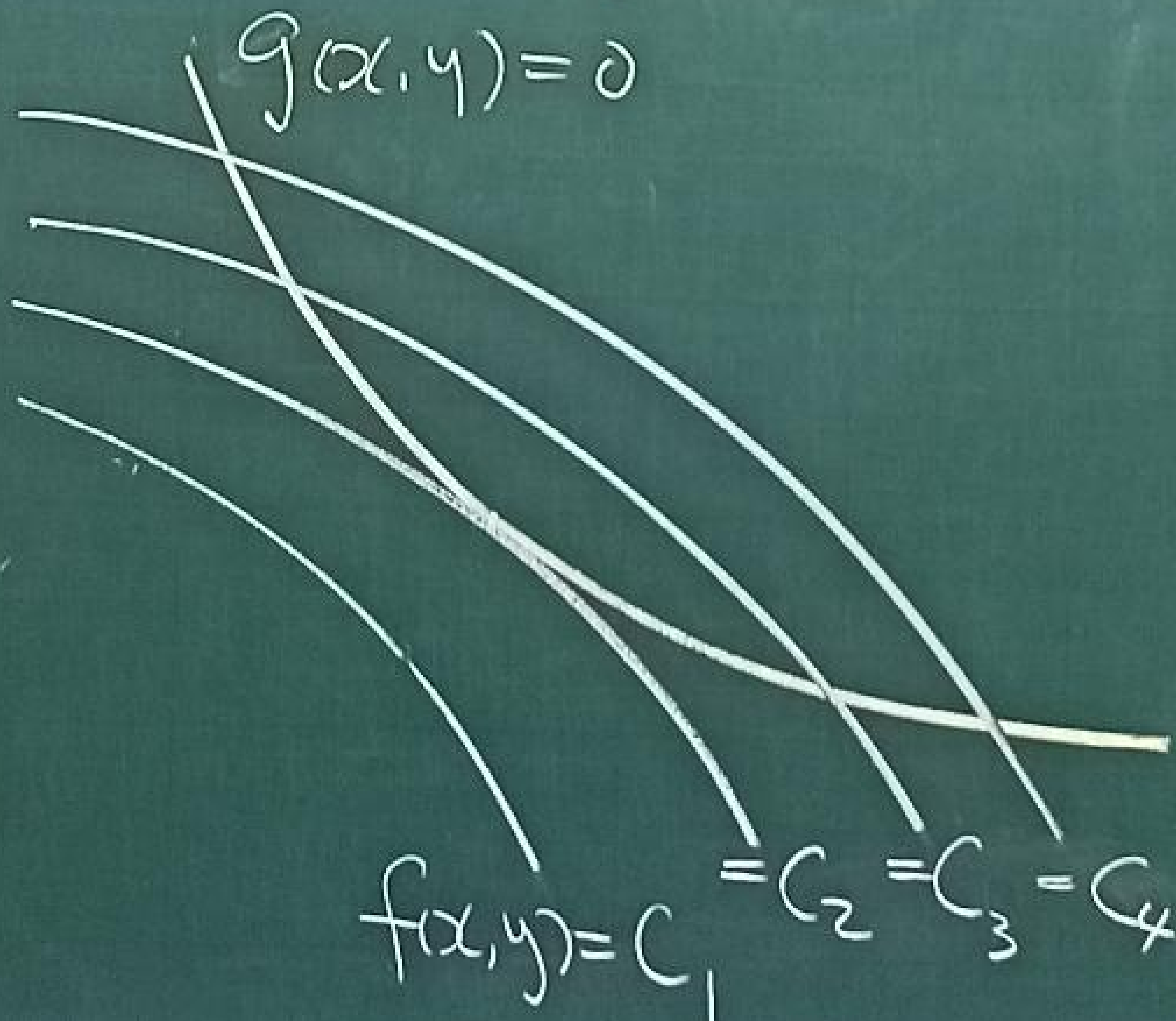
# Constrained Optimization
# (Lagrangian Multiplier)

<u>Goal</u>: find local min/max of $f(x,y)$ subject to the constraint $g(x,y)=0$



Local extreme at $(x_0, y_0)$

$\Rightarrow$ $\{g(x,y)=0\}$ and
$\not\Leftarrow$ $\{f(x,y)=c\}$ are
tangent at $(x_0, y_0)$

$\Leftrightarrow$ $\nabla f(x_0, y_0) \parallel \nabla g(x_0, y_0)$

or $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

for some constant $\lambda$

$\Rightarrow$ Solve $(x_0, y_0)$, $\lambda$

from $\begin{cases} g(x_0, y_0) = 0 \\ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases}$

3 equations, 3 unknowns

Eg 1. Find the nearst point to origin on $(x-\frac{1}{2})^2+\frac{y^2}{4}=1$

Sol: minimize $f(x,y)=(x-0)^2+(y-0)^2$

subject to $(x-\frac{1}{2})^2+\frac{y^2}{4}=1$

$$\begin{cases} (1)\quad (x_0-\frac{1}{2})^2+\frac{y_0^2}{4}=1 & (g=0) \\ (2)\quad 2x_0=\lambda 2(x_0-\frac{1}{2}) & (f_x=\lambda g_x) \\ (3)\quad 2y_0=\lambda \frac{y_0}{2} & (f_y=\lambda g_y) \end{cases}$$

(3) $\Rightarrow y_0=0$ or $\lambda=4$

Case A: $y_0=0$

$$\begin{cases} (x_0-\frac{1}{2})^2=1 & (A1) \\ 2x_0=2\lambda(x_0-\frac{1}{2}) & (A2) \end{cases}$$

$(A_1) \Rightarrow x_0 = \frac{-1}{2}$ or $\frac{3}{2}$

$(A_2) \Rightarrow \lambda = \frac{1}{2} \quad 3$

<u>Case B</u> $\lambda = 4$

$(2) \Rightarrow 2x_0 = 8(x_0 - \frac{1}{2})$

$x_0 = \frac{2}{3}$

$(1) \Rightarrow y_0 = \frac{\pm\sqrt{35}}{3}$

Overall, we have

$(x_0, y_0) =$

$\left(\frac{-1}{2}, 0\right), \left(\frac{3}{2}, 0\right), \left(\frac{2}{3}, \frac{\pm\sqrt{35}}{3}\right)$

$f = \underset{\text{min}}{\frac{1}{4}}, \quad \frac{9}{4}, \quad \underbrace{\frac{13}{3}, \frac{13}{3}}_{\text{max}}$

Find local extremes of

$f(x, y, z)$ subject to

$g(x, y, z) = 0$

Solve for $(x_0, y_0, z_0)$, $\lambda$

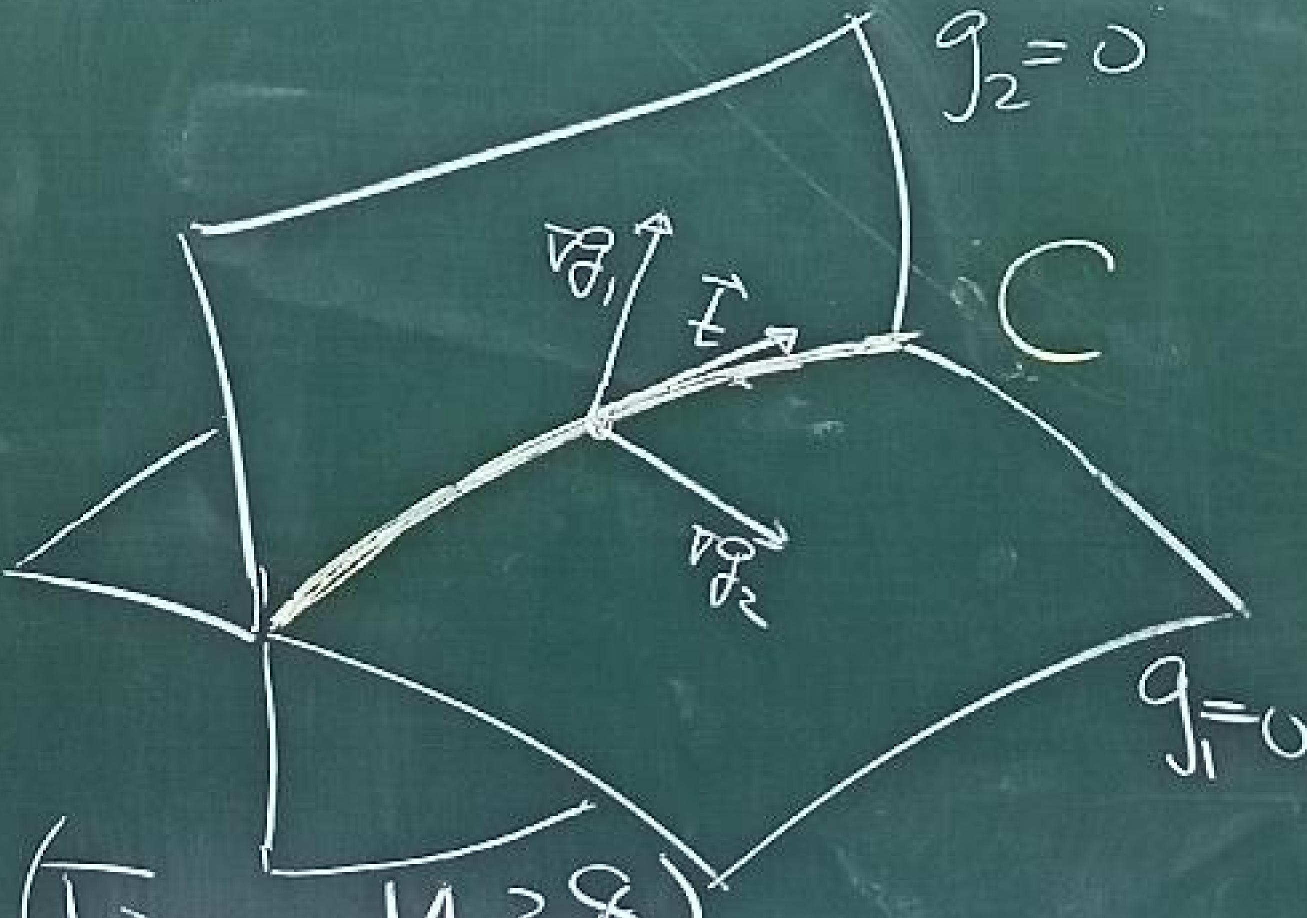
from

$$\begin{cases} g(x_0, y_0, z_0) = 0 \\ \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \end{cases}$$

4 unknowns, 4 equations.

Find local extremes of $f(x,y,z)$ subject to $\begin{cases} g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$

(Find extremes of $f$ on a curve $C$)




(Figure 14.38)

At a local extreme point

$\nabla g_1(x_0, y_0, z_0) \perp \vec{t}$ (tangent vector of $C$ at $(x_0, y_0, z_0)$)

($C \subseteq \{g_1 = 0\}$, $\vec{t} \perp$ normal vector of $\{g_1 = 0\}$)

Similarly

$\nabla g_2(x_0, y_0, z_0) \perp \vec{t}$

Moreover $\{f(x,y,z) = c\}$ is also tangent to $C$ (or to $\vec{t}$)

$\Rightarrow \nabla f(x_0, y_0, z_0) \perp \vec{t}$

$\therefore$ $\nabla f, \nabla g_1, \nabla g_2$ are <u>co-plane</u> at $(x_0, y_0, z_0)$

$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ at $(x_0, y_0, z_0)$

Summary:

Solve $x_0, y_0, z_0, \lambda_1, \lambda_2$ from

$$g_1(x_0, y_0, z_0) = 0$$

$$g_2(x_0, y_0, z_0) = 0$$

$$\begin{cases} \nabla f(x_0, y_0, z_0) \\ = \lambda_1 \nabla g_1(x_0, y_0, z_0) \\ \quad + \lambda_2 \nabla g_2(x_0, y_0, z_0) \end{cases}$$

$$\Vert$$

$$\partial_x f = \lambda_1 \partial_x g_1 + \lambda_2 \partial_x g_2$$

$$\partial_y f = \lambda_1 \partial_y g_1 + \lambda_2 \partial_y g_2$$

$$\partial_z f = \lambda_1 \partial_z g_1 + \lambda_2 \partial_z g_2$$

5 unknowns, 5 equations.