$$\Delta f = \Delta_1 + \Delta_2 + \Delta_3$$

$$\Delta f = f(x,y) - f(x_0,y_0)$$

$$\Delta_1 = \nabla f(x_0,y_0) \cdot (\Delta x, \Delta y)$$

$$\Delta_2 = \frac{1}{2}\left(A\,\Delta x^2 + 2B\,\Delta x \Delta y + C\,\Delta y^2\right)$$

$$A = f_{xx}(x_0,y_0),\ B = f_{xy}(x_0,y_0),\ C = f_{yy}(x_0,y_0)$$

$$|\Delta_3| \leq \frac{M}{3!}\left(|\Delta x| + |\Delta y|\right)^3$$

Def: (x_0,y_0) is a critical point of f if $\nabla f(x_0,y_0) = \vec{0}$ or $\nabla f(x_0,y_0)$ does not exist.

If f is differentiable at (x_0, y_0)
and $f(x_0, y_0)$ is a local min/max

$$\Rightarrow \nabla f(x_0, y_0) = 0 \,; \ \Delta_1 = 0$$

leading order term $= \Delta_2$

$$(*) \ A(\Delta x)^2 + 2B(\Delta x \Delta y) + C(\Delta y)^2$$

$$\left(= A\left(\Delta x - \frac{-B+\sqrt{B^2-AC}}{A}\Delta y\right) \cdot \left(\Delta x - \frac{-B-\sqrt{B^2-AC}}{A}\Delta y\right) \right)$$

$$= A \cdot \left(\left(\Delta x + \frac{B}{A}\Delta y\right)^2 - \frac{B^2-AC}{A^2}(\Delta y)^2 \right)$$

if $B^2 - AC > 0$

On the other hand

$(*) = A \cdot \begin{pmatrix} \text{Sum of} \\ \text{squares} \end{pmatrix}$

if $B^2 - AC < 0$

Suppose $(\Delta x, \Delta y) \neq (0,0)$

$$\begin{cases} \Delta_2 \text{ always} > 0 \Leftrightarrow \begin{cases} A > 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ always} < 0 \Leftrightarrow \begin{cases} A < 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ changes sign} \Leftrightarrow B^2 - AC > 0 \end{cases}$$

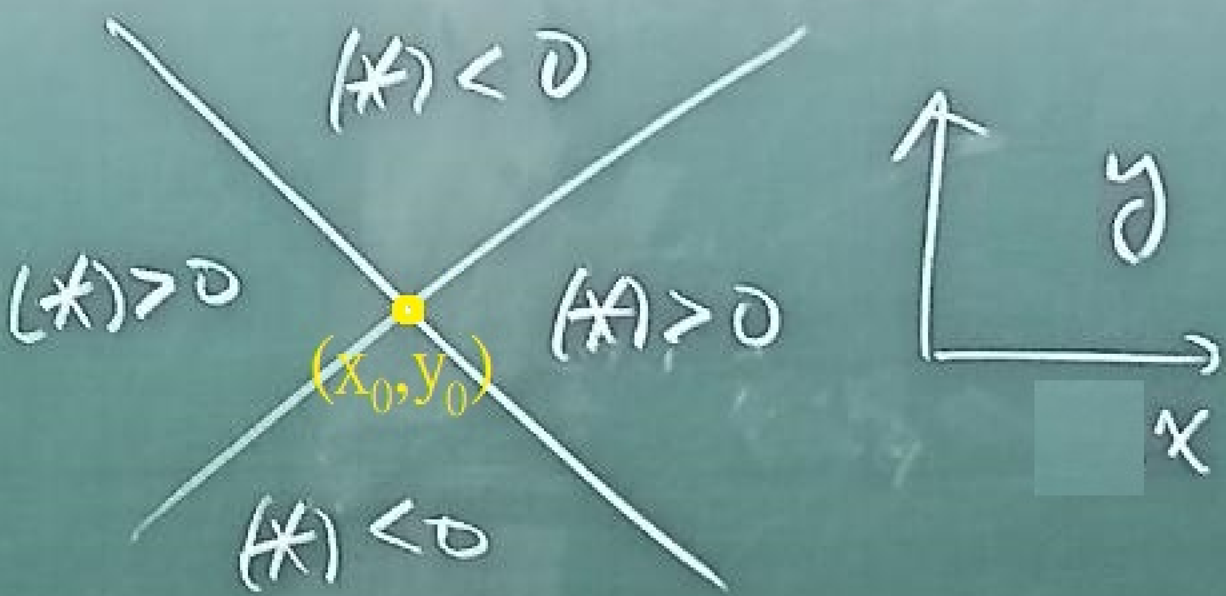
The Second Derivative Test:

$f_{xx} > 0$, $f_{xy}^2 - f_{xx} f_{yy} < 0$: local min

$f_{xx} < 0$, $f_{xy}^2 - f_{xx} f_{yy} < 0$: local max

$f_{xy}^2 - f_{xx} f_{yy} > 0$: Saddle point

If $B^2 - AC > 0$



Def: (x_0, y_0) is a saddle point of f

if $\begin{cases} \nabla f = (0,0) \\ f_{xy}^2 - f_{xx} f_{yy} > 0 \end{cases}$ at (x_0, y_0)

Summary: How to find

interior local extremes of f?

not on boundary (differentiable)

Step 1: Find all critical points where $\nabla f(x_0, y_0) = (0,0)$

Step 2: $D = (f_{xy}^2 - f_{xx} f_{yy})(x_0, y_0)$

(i) If $f_{xx}(x_0, y_0) < 0$, $D < 0$: local max

(ii) If $f_{xx}(x_0, y_0) > 0$, $D < 0$: local min

(iii) If $D > 0$: saddle point.

(iv) $D = 0$: no conclusion.

Eg1 find local extremes

of x^2+y^2-4y+9

Ans: Method 1:

$f_x=2x,\ f_y=2(y-2)$

$\Rightarrow$ critical point $=(0,2)$

$f_{xx}(0,2)=2,\ f_{xy}(0,2)=0$

$f_{yy}(0,2)=2 \Rightarrow D=-4$

$f_{xx}>0,\ D<0$

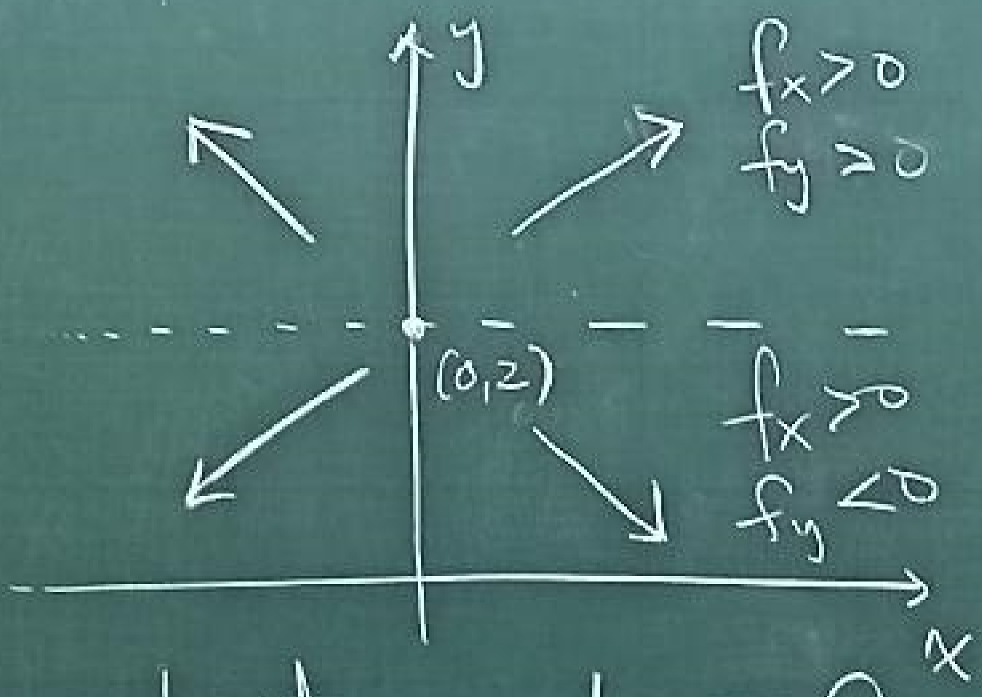
$\Rightarrow$ local min: $f(0,2)$

local max: none

Method 2: (Gradient Analysis)

$$f_x = 2x \begin{cases} >0, & x>0 \\ <0, & x<0 \end{cases}$$

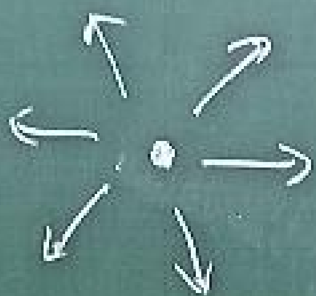
$$f_y = 2(y-2) \begin{cases} >0, & y>2 \\ <0, & y<2 \end{cases}$$



From directions of ∇f near $(x_0, y_0) = (0,2)$

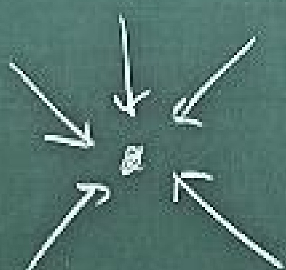
$\Rightarrow$ $\cancel{(x_0, y_0)}$ $f(0,2)$ is a local min

Remark. near a critical point



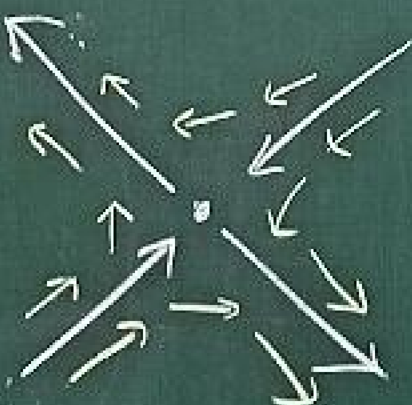
∇f points outward

$\Rightarrow$ local min



∇f points inward

$\Rightarrow$ local max



∇f points outward in some directions and inward in some directions: Saddle point

Find Absolute extremes

of $f(x,y) = 2+2x+2y-x^2-y^2$

on the region bounded by $\begin{cases} x=0 \\ y=0 \\ y=9-x \end{cases}$

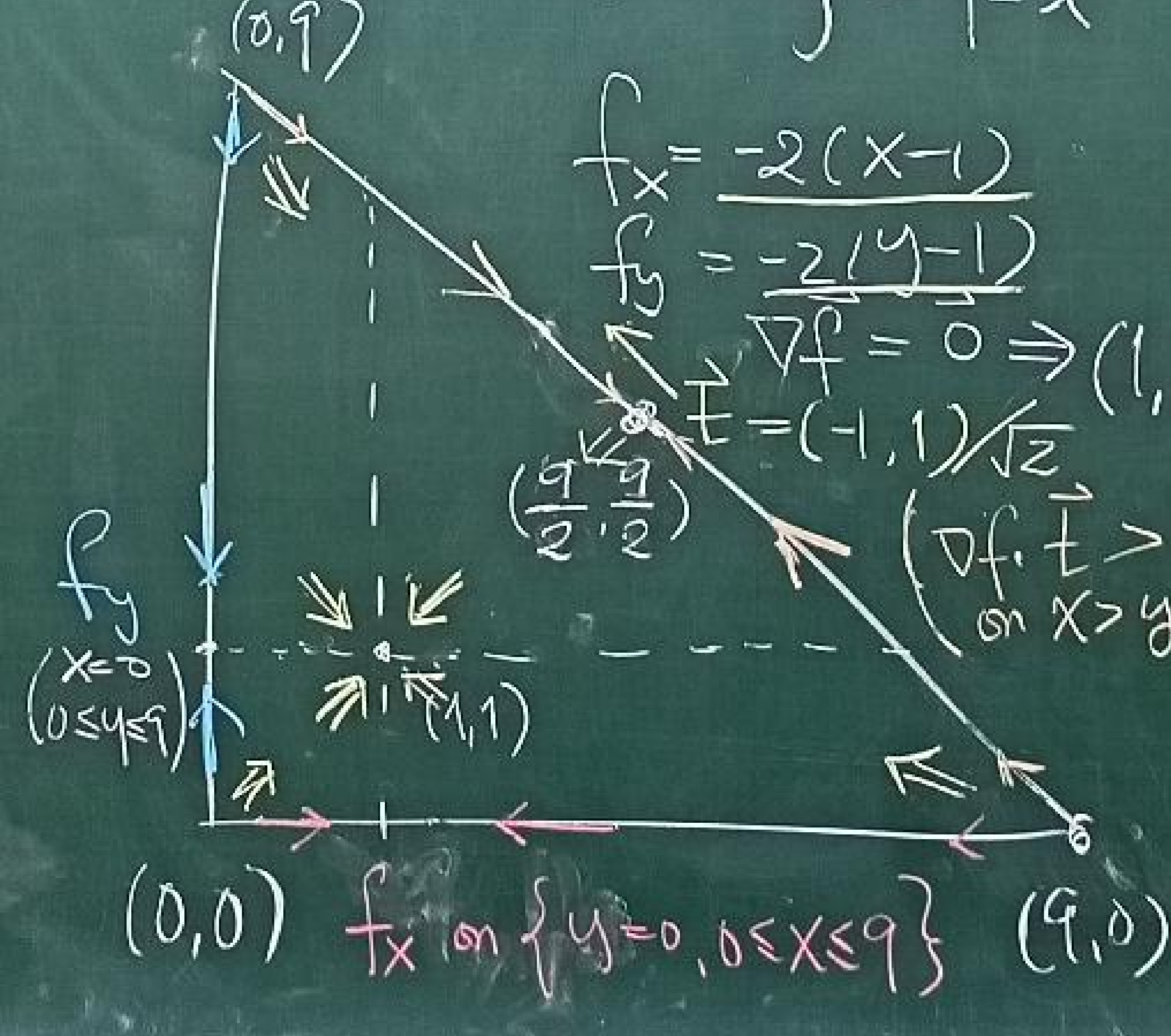
$f_x = -2(x-1)$

$f_y = -2(y-1)$

$\nabla f = 0 \Rightarrow (1,1)$

$\vec{t} = (-1,1)/\sqrt{2}$

$\left(\begin{array}{c} \nabla f \cdot \vec{t} > 0 \\ \text{on } x>y \end{array}\right)$



Method 1: (textbook)

critical point $= (1,1)$, compare $f(1,1)$ with all values of f on boundary

Method 2:

Near $(1,1)$: local max (and global max)

on $\begin{cases} y=0 \\ 0\le x\le 9 \end{cases}$ $(1,0)$ 1D local max

on $\begin{cases} x=0 \\ 0\le y\le 9 \end{cases}$ $(0,1)$ 1D local max

on $\begin{cases} x+y=9 \\ 0\le x\le 9 \end{cases}$. $\nabla f\cdot \vec{t} = \left(-2(x-1),-2(y-1)\right)\cdot(-1,1)/\sqrt{2}$

$= \sqrt{2}(x-y)\begin{cases} >0 & x>y \\ <0 & x<y \end{cases}$

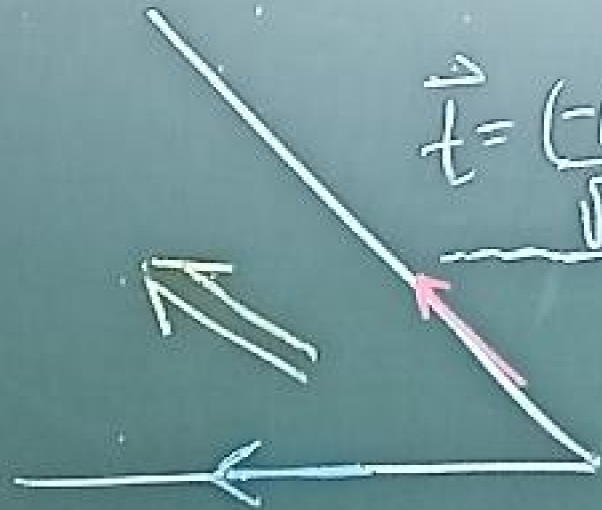
$\vec{t}$

$\Rightarrow$ $x<y$ $x>y$

Absolute max $= f(1,1)$

Absolute min: Compare $f(0,0)$, $f(0,9)$, $f(9,0)$

Rm



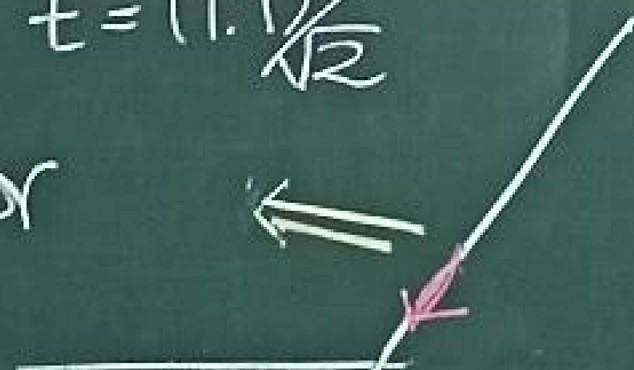
$\vec{t} = \frac{(-1,1)}{\sqrt{2}}$

$(f_x < 0,\ f_y > 0)$

$$\Rightarrow \begin{cases} \left(\nabla f \cdot \frac{(-1,1)}{\sqrt{2}} > 0\right) \\ \left(\nabla f \cdot (-1,0) > 0\right) \end{cases}$$

$\vec{t} = (1,1)/\sqrt{2}$

or



$$f_x < 0,\ f_y > 0 \Rightarrow \left\{ \text{or} \right. \quad \left(\nabla f \cdot \frac{(1,1)}{\sqrt{2}} \gtrless 0\right)$$

Computing $\nabla f \cdot \vec{t}$ on bdry is necessary here