Linearization (linear approximation)

If $f(x,y)$ is differentiable at (x_0, y_0), then

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

$$\left(\text{or } f(x,y)-L(x,y) = \begin{matrix} \varepsilon\sqrt{(x-x_0)^2+(y-y_0)^2} \\ \text{or } \varepsilon_1(x-x_0)+\varepsilon_2(y-y_0) \end{matrix} \right)$$

$\Rightarrow$ $f(x,y)$ is close to $L(x,y)$ near (x_0,y_0)

"linearization": use $L(x,y)$ to approximate $f(x,y)$

Eg 1. $f(x,y) = x^2 - xy + \frac{y^2}{2} + 3$

Find approximate value of $f(3.1, 1.9)$

<u>Sol</u>: Take $(x_0, y_0) = (3, 2)$

$L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)$

approximation $= L(3.1, 1.9)$

$f(3,2) = 8$, $f_x(3,2) = 4$, $f_y(3,2) = -1$

$\therefore$ Ans $= 8 + 4\cdot(0.1) - 1\cdot(-0.1) = 8.5$

<u>Rm</u>: $\underset{f(x,y) - f(x_0,y_0)}{\underset{\parallel}{\Delta z}} \cong f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$

$\Rightarrow f(x,y) \cong L(x,y)$

Error of linear approximation

$f(x,y) \cong L(x,y)$

How large is $|f(x,y) - L(x,y)|$?

$y = f(x)$ case

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{L(x)} + \frac{f''(c)}{2}(x-x_0)^2$$

$z = f(x,y)$ case.

$$f(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{L(x,y)}$$

$$\underbrace{+ \frac{1}{2}\left(f_{xx}(c_1,c_2)(x-x_0)^2 + 2f_{xy}(c_1,c_2)(x-x_0)(y-y_0) + f_{yy}(c_1,c_2)(y-y_0)^2 \right)}_{\text{error}}$$

(pf: next week)

Taylor's formula on p883, n=1

Corollary: If $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are all continuous in an open region R and $(x_0, y_0) \in R$.

and $|f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$ on R

Then:

$$|f(x,y) - L(x,y)| \leq \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2$$

error estimate for linear app.

In Eg 1, $f_{xx} = 2$, $f_{xy} = -1$, $f_{yy} = 1$

$\therefore M = 2$ will do.

$$|f(3.1, 1.9) - L(3.1, 1.9)| \leq \frac{2}{2}(0.1 + 0.1)^2$$

$$= 0.04$$

Eg 2 Volume of Cylinder

$$V(r,h) = \pi r^2 h$$

If $r_0 = 1$, $h_0 = 5$.

$r_1 = r_0 + \Delta r$, $\Delta r = 0.1$

$h_1 = h_0 + \Delta h$, $\Delta h = -0.1$

$$\Delta V = V(r_1, h_1) - V(r_0, h_0) = ?$$

Sol: $\Delta V \cong V_r(r_0,h_0)\Delta r + V_h(r_0,h_0)\Delta h$

$$= 2\pi r_0 h_0 \Delta r + \pi r_0^2 \Delta h = 0.9\pi.$$

Error: $V_{rr} = 2\pi h$, $V_{rh} = 2\pi r$, $V_{rr} = 0$

they are evaluated at (c_1, c_2) (between (r_0, h_0) and (r_1, h_1))

$|V_{rr}(c_1,c_2)| = 2\pi c_2 \leq \max(2\pi h_0, 2\pi h_1) = 2\pi \cdot 5$ $= M$

$|V_{rh}(c_1,c_2)| = 2\pi c_1 \leq \max(2\pi r_0, 2\pi r_1) = 2\pi \cdot 1.1$

$$|Err| \leq \frac{10\pi}{2}(0.1+0.1)^2 = 0.2\pi$$

Extreme Values and Saddle points

How to find local min/max of $f(x,y)$ (assuming all derivatives are cont.)

(1) At a local min/max (x_0, y_0)

$\Rightarrow$ $\nabla f(x_0, y_0) = (0,0)$ First Derivative Test
$\not\Leftarrow$

i.e. the tangent plane is horizontal

$\because \Delta z = \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y)$

If $\nabla f(x_0, y_0) \neq (0,0)$

then $\Delta z > 0$ (< 0) if $(\Delta x, \Delta y) = \nabla f / |\nabla f|$ $(-\nabla f / |\nabla f|)$

$\therefore$ $L(x,y)$ (and $f(x,y)$) increases (decreases) in the direction of $\nabla f / |\nabla f|$ $(-\nabla f / |\nabla f|)$

It can be shown that p883, Taylor's formula, n = 2

$$\Delta f = \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y) \quad \cdots \Delta_1$$

$$+ \frac{1}{2}\left(f_{xx}(x_0, y_0)(\Delta x)^2 + 2 f_{xy}(x_0, y_0)\Delta x \Delta y + f_{yy}(x_0, y_0)(\Delta y)^2 \right) \cdots \Delta_2$$

$$+ \text{"}O\text{"}\left((\Delta x)^3, (\Delta x)^2 \Delta y, \Delta x (\Delta y)^2, (\Delta y)^3 \right) \cdots \Delta_3$$

See "Big O" on p464

$\Delta_{1,2,3} =$ (linear, quadratic, cubic) in $\Delta x, \Delta y$

At a critical point, $\nabla f = 0$, $\therefore \Delta_1 = 0$

leading order term $= \Delta_2$

$$\Delta_2 = A(\Delta x)^2 + 2B \Delta x \Delta y + C(\Delta y)^2$$

$$A = f_{xx}(x_0, y_0),\ B = f_{xy}(x_0, y_0),\ C = f_{yy}(x_0, y_0)$$

$(\Delta_3 \ll \Delta_2$ if $\Delta_2 \neq 0)$