Properties of ∇f

If f is differentiable at (x_0, y_0)

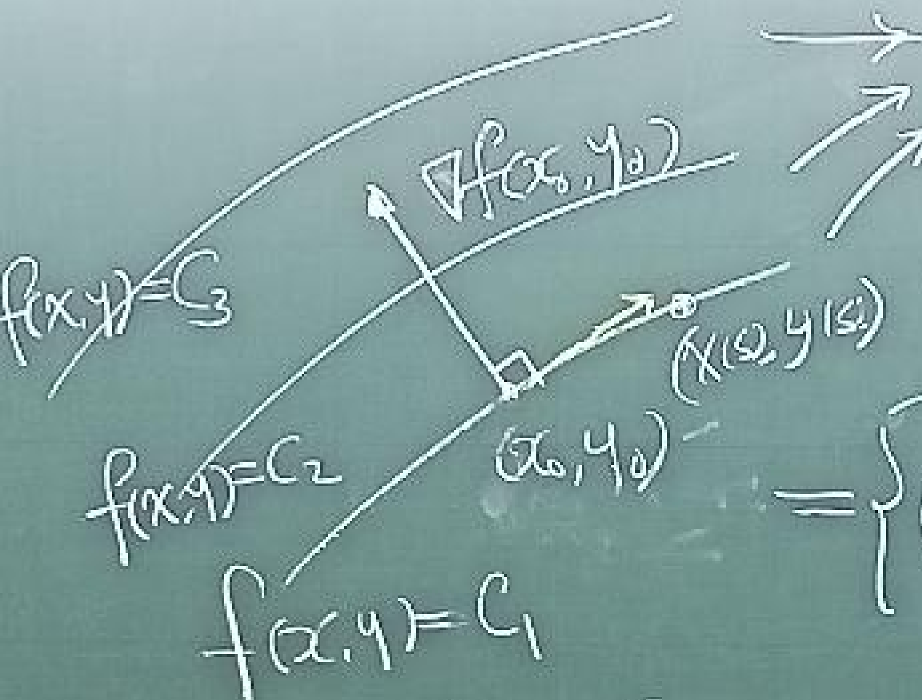
$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

$$= |\nabla f(x_0, y_0)| \cdot |\vec{u}| \cdot \cos\theta$$

$\Rightarrow$ (1) f increases (decreases) most rapidly in the direction of $\vec{u}$, if $\cos\theta = 1$ ($\cos\theta = -1$)

i.e. in the direction of ∇f $(-\nabla f)$

(2) $D_{\vec{u}} f(x_0, y_0) = 0 \Leftrightarrow \nabla f(x_0, y_0) \perp \vec{u}$



level curves of $f(x,y)$

$$=\{(x,y) \mid f(x,y)=C\}$$

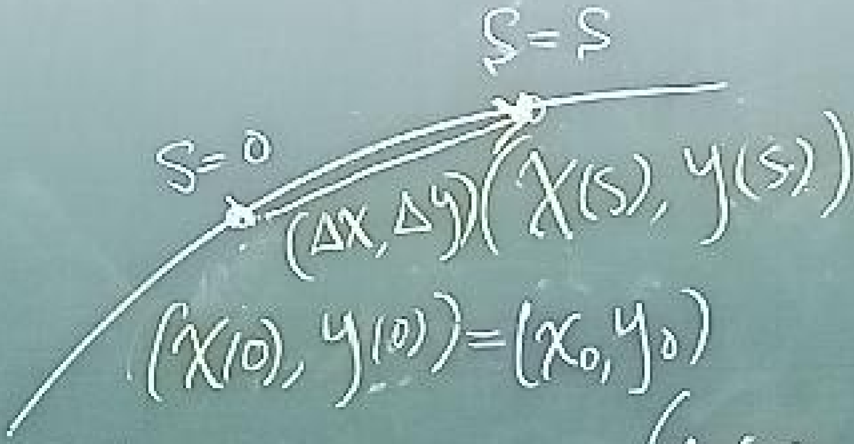
Let $(x(s),y(s))$ be a level curve of f (differentiable)

$$\Rightarrow f(x(s),y(s)) = \underset{(\text{in } s)}{\text{constant}}$$

$$\frac{d}{ds} \Rightarrow \nabla f(x(s),y(s))\cdot(x'(s),y'(s))=0$$

$(x'(0),y'(0))$ = a tangent vector

$$\therefore \nabla f(\underset{\substack{\| \\ (x(0),\,y(0))}}{x_0,y_0}) \perp \begin{array}{c}\text{tangent line of} \\ \{(x,y) \mid f(x,y)=f(x_0,y_0)\}\end{array}$$



$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \lim_{s \to 0} \begin{pmatrix} \dfrac{x(s)-x(0)}{s-0} \\ \dfrac{y(s)-y(0)}{s-0} \end{pmatrix}$$

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

= a tangent vector

i.e. $(x'(0), y'(0))$ points in the direction of tangent line at $(x(0), y(0))$

Eg1: Find the tangent / normal lines of $\frac{x^2}{4}+y^2=2$ at $(-2,1)$

<u>Sol</u>: Let $f(x,y)=\frac{x^2}{4}+y^2$

$\therefore \frac{x^2}{4}+y^2=2$ is a level curve of f

tangent line

$(x+2,\ y-1)\cdot \underbrace{\nabla f(-2,1)}_{(-1,\,2)}=0$

i.e. $\underline{-1(x+2)+2(y-1)=0}$

Normal line $((\Delta x,\Delta y)\,/\!/\,\nabla f)$

$$\frac{y-1}{x+2}=\frac{2}{-1}$$

Remark:

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla(C f(x,y)) = c \nabla f(x,y)$

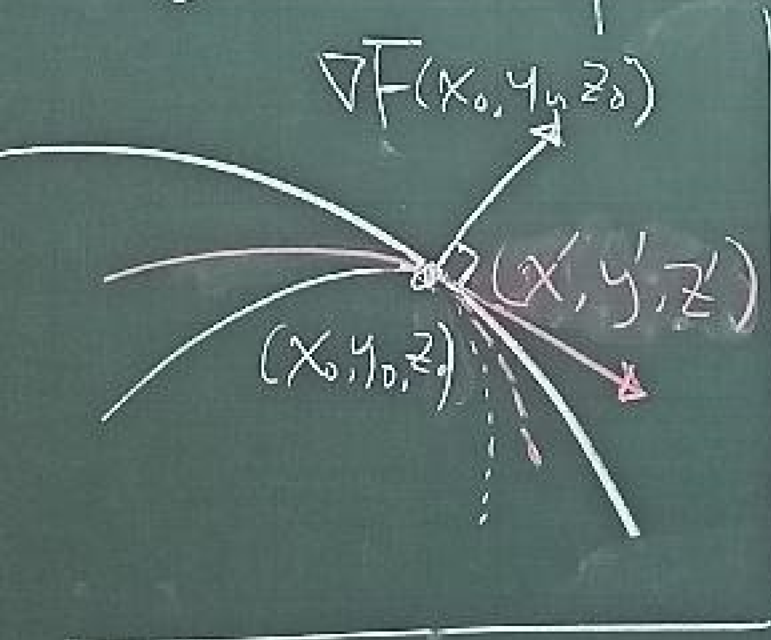
(iii) $\nabla(f \cdot g) = f \nabla g + g \nabla f$

(iv) $\nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$

Pf: Check x-component and y-component directly

Tangent plane and normal line of (differentiable) level surface of $F(x,y,z)$

$= \{(x,y,z) \mid F(x,y,z)=C\}$



Suppose $\{(x(t),y(t),z(t))\} \subset$ level surface of F at $(x_0,y_0,z_0)=(x(t_0),y(t_0),z(t_0))$

$$\Rightarrow F(x(t),y(t),z(t)) = \underbrace{F(x(t_0),y(t_0),z(t_0))}_{\text{constant}}$$

$$\left.\frac{d}{dt}\right|_{t_0} \Rightarrow \nabla F(x(t_0),y(t_0),z(t_0)) \cdot (x'(t_0),y'(t_0),z'(t_0)) = 0$$

Since $(x'(t_0), y'(t_0), z'(t_0))$

is a tangent vector

of the leve surface at (x_0, y_0, z_0)

$$\Rightarrow \nabla F(x_0, y_0, z_0) \perp \begin{pmatrix} \text{any tangent} \\ \text{vector at} \\ (x_0, y_0, z_0) \end{pmatrix}$$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \perp \begin{pmatrix} \text{tangent plane} \\ \text{at } (x_0, y_0, z_0) \end{pmatrix}$$

$\therefore$ Equation of normal line.

$$(x - x_0, y - y_0, z - z_0) \parallel \nabla F(x_0, y_0, z_0)$$

i.e. $$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

or
$$x(t) = x_0 + F_x(x_0, y_0, z_0)\, t$$
$$y(t) = y_0 + F_y(x_0, y_0, z_0)\, t$$
$$z(t) = z_0 + F_z(x_0, y_0, z_0)\, t$$

Equation of tangent line

$$(x-x_0, y-y_0, z-z_0)\cdot \nabla F(x_0,y_0,z_0)=0$$

Eg 2 Find the tangent plane and normal line of $x^2+\frac{y^2}{4}+\frac{z^2}{9}=3$ at $(1,2,3)$.

Sol. Let $F(x,y,z)=x^2+\frac{y^2}{4}+\frac{z^2}{9}$

$$\Rightarrow \nabla F(1,2,3)=\left(2x, \frac{y}{2}, \frac{2z}{9}\right)\Big|_{(1,2,3)}$$

$$=\left(2,1,\frac{2}{3}\right)$$

tangent plane:

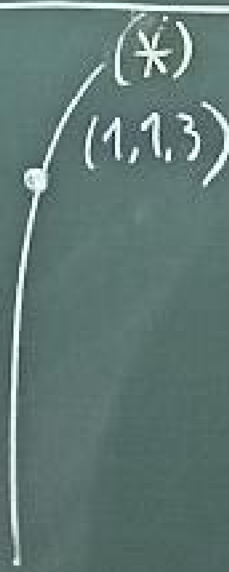
$$(x-1, y-2, z-3)\cdot\left(2,1,\frac{2}{3}\right)=0$$

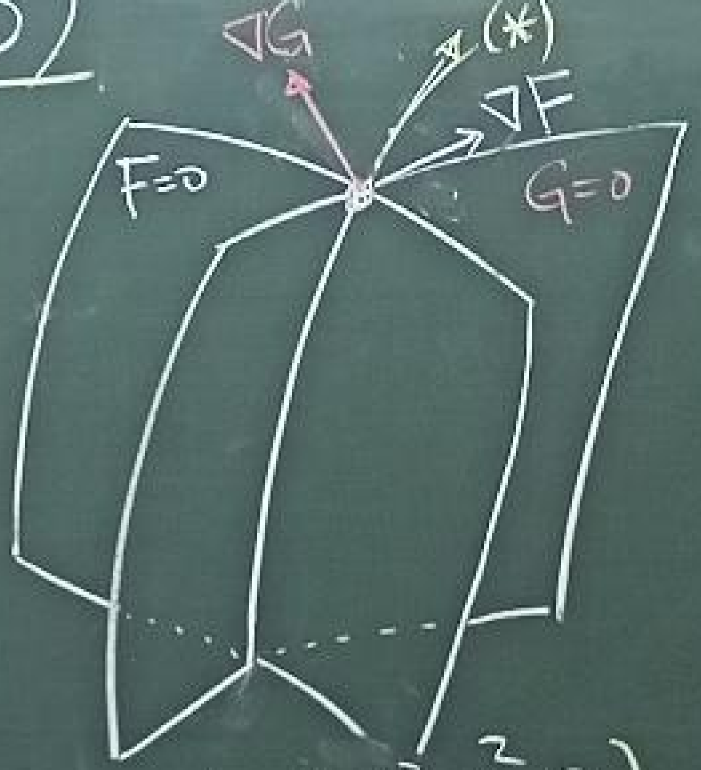
normal line: $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2/3}$

or $x(t) = 1+2t,\ y(t) = 2+t,$

$z(t) = 3 + (2t/3)$

Eg3 Find the tangent line of $\begin{cases} x^2+y^2-2=0 \\ x+z-4=0 \end{cases}$ (*) at $(1,1,3)$





General case

$$C=\left\{(x,y,z),\ \begin{matrix} \overset{(=x^2+y^2-2)}{F(x,y,z)}=0 \\ \underset{(=x+z-4)}{G(x,y,z)}=0 \end{matrix}\right\}$$

$(x_0,y_0,z_0)\in C$

$\vec{t}$: tangent vector of C at (x_0,y_0,z_0)

Since $C \subset \{F(x,y,z) = F(x_0,y_0,z_0)\}$
level curve of F

$\Rightarrow \vec{t} \perp \nabla F(x_0,y_0,z_0)$
(tangent vector of C)

Similarly $\vec{t} \perp \nabla G(x_0,y_0,z_0)$

Since $C \subset \{G(x,y,z) = G(x_0,y_0,z_0)\}$

$$\Rightarrow \vec{t} \parallel \underbrace{\nabla F(x_0,y_0,z_0) \times \nabla G(x_0,y_0,z_0)}$$

$$\left(\begin{vmatrix} i & j & k \\ 2x & 2y & 0 \\ 1 & 0 & 1 \end{vmatrix} =\right) \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix} \begin{array}{l} \perp \nabla F \\ \perp \nabla G \end{array}$$

Ans: $$\frac{x-1}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} = 2} = \frac{y-1}{\underset{=-2}{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}} = \frac{z-3}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} = -2}$$

or $x(t) = 1+2t$, $y(t) = 1-2t$, $z(t) = 3-2t$

Remark

$\vec{w} = (w_1, w_2, w_3)$

$\vec{u} = (u_1, u_2, u_3)$

$\vec{v} = (v_1, v_2, v_3)$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \cdots (*)$$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Then $(\vec{u} \times \vec{v}) \cdot \vec{w}$

$$= \left(w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + w_2 \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

$$= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$$ Replace $\vec{i}, \vec{j}, \vec{k}$ by w_1, w_2, w_3 in $(*)$

Corollary:

$$\vec{u} \times \vec{v} \cdot \vec{u} = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

$$\vec{u} \times \vec{v} \cdot \vec{v} = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

$$\Rightarrow \vec{u} \times \vec{v} \perp \vec{u}$$

$$\vec{u} \times \vec{v} \perp \vec{v}$$