Recall: If $z(x,y)$ is implicitly defined by

$(*)$ $F(x,y,z)=0$ and

$F(x_0,y_0,z_0)=0$. Then

$$\frac{\partial z}{\partial x}(x_0,y_0,z_0)=\frac{-\partial_x F(x_0,y_0,z_0)}{\partial_z F(x_0,y_0,z_0)}$$

$$\frac{\partial z}{\partial y}(x_0,y_0,z_0)=\frac{-\partial_y F(x_0,y_0,z_0)}{\partial_z F(x_0,y_0,z_0)}$$

Pf: $\frac{\partial}{\partial x}(*) \Rightarrow \partial_x F + \partial_z F \frac{\partial z}{\partial x} = 0$

$\frac{\partial}{\partial y}(*) \Rightarrow \partial_y F + \partial_z F \cdot \frac{\partial z}{\partial y} = 0$

Eg 1: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where

$F(x,y,z) = xy + z^3x - 2yz$

and $(x_0, y_0, z_0) = (1, 1, 1)$.

Ans:

$$\partial_x F(1,1,1) = y + z^3 \Big|_{(1,1,1)} = 2$$

$$\partial_y F(1,1,1) = x - 2z \Big|_{(1,1,1)} = -1$$

$$\partial_z F(1,1,1) = 3z^2x - 2y \Big|_{(1,1,1)} = 1$$

$$\therefore \frac{\partial z}{\partial x}(1,1,1) = \frac{-2}{1} = -2$$

$$\frac{\partial z}{\partial y}(1,1,1) = \frac{-(-1)}{1} = 1$$

<u>Def</u>: (Directional derivative)

Derivative of $f(x,y)$ at $P_0=(x_0,y_0)$ in the direction of the <u>unit</u> vector $\vec{u}=(u_1,u_2)$ $(u_1^2+u_2^2=1)$:

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} \left(= D_{\vec{u}} f(x_0,y_0)\right)$$

$$\stackrel{\text{def}}{=\!=} \lim_{s\to 0} \frac{f(x_0+su_1,\, y_0+su_2)-f(x_0,y_0)}{s-0}$$

$$\left(= \lim_{s\to 0} \frac{f(x(s),y(s))-f(x(0),y(0))}{s-0}\right.$$

$$\left.\text{where } x(s)=x_0+su_1,\ y(s)=y_0+su_2\right)$$

Thm: If $f(x,y)$ is differentiable at (x_0, y_0),

Then $D_{\vec{u}} f(x_0,y_0) = \overset{\text{gradient}}{\nabla} f(x_0,y_0) \cdot \vec{u}$

where $\nabla f = (f_x, f_y)$

pf: $\Delta z \left(= f(x,y) - f(x_0,y_0) \right)$

$= f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

Let $x = x_0 + \Delta s u_1$, $y = y_0 + \Delta s u_2$

$\Delta x = x - x_0 = \Delta s u_1$, $\Delta y = y - y_0 = \Delta s u_2$,

$\Delta s = s - 0$

$\Rightarrow D_{\vec{u}} f(x_0,y_0) = \lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s}$

$= \lim_{\Delta s \to 0} \left(f_x(x_0,y_0)\frac{\Delta x}{\Delta s} + f_y(x_0,y_0)\frac{\Delta y}{\Delta s} \right) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2$

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Eg 2: $f(x,y) = x^2 + xy$

(f is differentiable)

$\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$D_{\vec{u}} f(1,2) = ?$

$$\text{Ans} = \nabla f(1,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{f_x(1,2)}{\sqrt{2}} + \frac{f_y(1,2)}{\sqrt{2}}$$

$$f_x = 2x + y, \quad f_y = x$$

$$\text{Ans} = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

Eg 3. $f(x,y)=\begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Evaluate $D_{\vec{u}}f(0,0)$ and $\nabla f(0,0)\cdot\vec{u}$

where $\vec{u}=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ [See Fig 14.13]

<u>Ans</u>. $f_x(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0}=0$

$$f_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=0$$

$$D_{\vec{u}}f(0,0)=\lim_{s\to 0}\frac{f\left(\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}}\right)-f(0,0)}{s-0}$$

$$\lim_{s\to 0}\frac{\left(\frac{s^3}{2\sqrt{2}}\right)/s^2}{s-0}=\frac{1}{2\sqrt{2}}\neq\nabla f\cdot\vec{u}=0$$

(Corollary: f is not differentiable at $(0,0)$)

In contrast, if

$f(x,y) = ax+by+c$ (plane)

(f is a linear function)

$\Rightarrow f_x(0,0) = a, \; f_y(0,0) = b$

$$D_{\vec{u}} f(0,0) = \lim_{s \to 0} \frac{(asu_1 + bsu_2 + c) - c}{s}$$

$$= au_1 + bu_2 = \nabla f(0,0) \cdot \vec{u}$$

i.e. a linear function satisfies (#)

Tip: f has a tangent plane at (x_0, y_0)

$\Rightarrow f$ is "close to" a ~~plane~~ linear function near (x_0, y_0)

$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$ (#)

A linear function satisfies (#), so does a differentiable fun.

But the function in Eg3 does not!