Def: $f(x,y)$ is differentiable at (x_0, y_0) if

(i) $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist

(*)(ii)

$$f(x,y) = \underbrace{\boxed{f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)}}_{= L(x,y)} + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0)$$

where $\lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon_1 = 0 = \lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon_2$

Same definition as next page

Def: $f(x,y)$ is diff. at (x_0,y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\Delta z = f(x,y) - f(x_0,y_0)$

$\Delta x = x - x_0$

$\Delta y = y - y_0$

and $\lim_{(\Delta x,\Delta y)\to(0,0)} (\varepsilon_1,\varepsilon_2) = (0,0)$

Same definition as last page

Remark:

(a) We can replace

$$\varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y$$

by $\varepsilon \cdot \sqrt{(\Delta x)^2 + (\Delta y)^2}$

where $\lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon = 0$

Eg 1: $(\Delta x)^2 + (\Delta y)^2$

$$= \underset{\overset{\parallel}{\varepsilon_1}}{\underline{\Delta x}} \cdot \Delta x + \underset{\varepsilon_2}{\underline{\Delta y}} \cdot \Delta y$$

$$= \underbrace{\sqrt{(\Delta x)^2 + (\Delta y)^2}}_{\varepsilon} \cdot \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

For general ε case, see homework 07

(b) Recall

$z = f(x,y)$ and $z = g(x,y)$

are tangent at $(x_0, y_0, f(x_0,y_0))$

if $\displaystyle \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-g(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

That is

[distance between $f(x,y)$ and $g(x,y)$]

is much smaller than

[distance between (x,y) and (x_0,y_0)]

Therefore $(*)+(a)$ simply means

$z=f(x,y)$ and $z=L(x,y)$ are tangent at $(x_0, y_0, f(x_0, y_0))$

That is $z=L(x,y)$ is the tangent plane of $z=f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$

$)]$

$_0, y_0)]$

Remark If $\exists\, a, b \in \mathbb{R}$

such that $y = f(x)$

is tangent to $y = a(x - x_0) + b$ at $(x_0, f(x_0))$

then (i) f is diff. at x_0

(ii) $a = f'(x_0)$, $b = f(x_0)$

(i.e. need not assume $f'(x_0)$ exist)

Pf

$$\lim_{x \to x_0} \frac{f(x) - (a(x - x_0) + b)}{x - x_0} = \underbrace{\lim_{x \to x_0} \frac{f(x) - b}{x - x_0}}_{= \frac{f(x_0) - b}{0}} - a = 0$$

exist $\Rightarrow \begin{cases} f(x_0) - b = 0 \\ \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} - a = 0 \end{cases}$

Thm 4: $f(x,y)$ is diff. at (x_0,y_0) $(*_1)$

$\Longrightarrow f(x,y)$ is cont. at (x_0,y_0) $(*_2)$

pf: (Important)

$$(*_2) \Longleftrightarrow \overbrace{f(x,y)-f(x_0,y_0)}^{\Delta z} = \varepsilon_a \quad \left(\lim \varepsilon_a = 0\right)$$

i.e. $$\lim_{(x,y)\to(x_0,y_0)} f(x,y)-f(x_0,y_0) = 0$$

$$(*_1) \Longleftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2 \Delta y$$

Since
$$\lim_{(x,y)\to(x_0,y_0)} (f_x+\varepsilon_1)\Delta x + (f_y+\varepsilon_2)\Delta y = 0$$

$\therefore (*_1) \Rightarrow (*_2)$

When is f diff. at (x_0, y_0)?

Note: $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist

$\not\Rightarrow$ f is cont. at (x_0, y_0)

$\not\Rightarrow$ f is diff. at (x_0, y_0)

Thm3: R is an open region $(x_0, y_0) \in R$. If f, f_x, f_y are defined on R and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0)

Proof: See Appendix 9 of the textbook

See homework 07 for more details

Eg2 $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

Is $f(x,y)$ differentiable?

Ans: (i) f is cont. on $\mathbb{R}^2$ (including $(0,0)$)

the x-y plane

(ii) f_x, f_y are cont. at $\mathbb{R}^2\setminus(0,0)$

$\overset{}{\underset{\text{Thm 3}}{\Longrightarrow}}$ f is diff. on $\mathbb{R}^2\setminus(0,0)$

$f_x=\dfrac{y^3}{(x^2+y^2)^{\frac{3}{2}}}$, $(x,y)\neq(0,0)$

$f_x(0,0)=\lim\limits_{x\to0}\dfrac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to0}\dfrac{0-0}{x}=0$

$\lim\limits_{x\to0}f_x(x,mx)=\dfrac{m^3}{(1+m^2)^{\frac{3}{2}}}\xrightarrow{\text{2 path}}\lim\limits_{(x,y)\to(0,0)}f_x$ does not exist

$\Rightarrow$ Thm 3 not applicable at $(0,0)$

Proof of Theorem 2: Appendix 9 of the textbook

Thm 2: If f, f_x, f_y, f_{xy}, f_{yx} are cont. in an open region R and $(x_0, y_0) \in R$

Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

$$\underset{\partial_y(\partial_x f(x_0,y_0))}{f_{xy}(x_0, y_0)} = \underset{\partial_x(\partial_y f(x_0,y_0))}{f_{yx}(x_0, y_0)}$$

Remark. $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Then $f_{xy} = f_{yx}$ on $\mathbb{R}^2 \setminus (0,0)$

but $f_{xy}(0,0) \neq f_{yx}(0,0)$

See section 14.3, problem 72 for details

(Note: f_{xy} and f_{yx} are not cont. at (x_0, y_0))

Chain Rule

Single variable

$$\frac{d}{dx} f(g(x)) = \left.\frac{df}{dy}\right|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Two variables $z = f(x, y)$

$$\left.\frac{d}{dt} f(x(t), y(t))\right|_{t=t_0} = \lim_{t \to t_0} \frac{\Delta z}{\Delta t}$$

(assume f is differentiable)

$$= \lim_{\Delta t \to 0} \left(f_x(x(t), y(t)) + \varepsilon_1\right) \frac{\Delta x}{\Delta t} + \left(f_y(x(t), y(t)) + \varepsilon_2\right) \frac{\Delta y}{\Delta t}$$

$$= f_x(x(t_0), y(t_0))\, x'(t_0) + f_y(x(t_0), y(t_0))\, y'(t_0)$$

(assume $x(t)$, $y(t)$ are differentiabl

Similarly

$$\partial_s f(x(s,t), y(s,t), z(s,t))$$

$$= f_x(x(s,t), y(s,t), z(s,t)) \cdot \partial_s x(s,t)$$

$$+ f_y(\qquad\qquad) \cdot \partial_s y(s,t)$$

$$+ f_z(\qquad\qquad) \cdot \partial_s z(s,t)$$

Similarly

$$\partial_t f(\qquad\qquad)$$

$$= f_x(\qquad\qquad) \partial_t x(s,t)$$

$$+ f_y(\qquad\qquad) \partial_t y(s,t)$$

le) $$+ f_z(\qquad\qquad) \partial_t z(s,t)$$

Implicit differentiation revisited.

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x,y,z)=0$

<u>Sol.</u> $F(x,y,z(x,y)) = 0$

$$\partial_x F(x,y,z(x,y))$$

fixed.

$$= \partial_1 F \cdot \underset{=1}{\frac{\partial x}{\partial x}} + \partial_2 F \underset{=0}{\frac{\partial y}{\partial x}} + \partial_3 F \cdot \frac{\partial z}{\partial x}$$

$$= \partial_1 F(x,y,z(x,y)) \cdot$$
$$+ \partial_3 F(x,y,z(x,y)) \cdot \frac{\partial z}{\partial x}$$

$$\Rightarrow \frac{\partial z}{\partial x}(x,y,z) = -\frac{\partial_1 F(x,y,z)}{\partial_3 F(x,y,z)}$$
$$= -\frac{\partial_x F(x,y,z)}{\partial_z F(x,y,z)}$$