Def: f is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

(Recall ε-δ definition from last lecture)

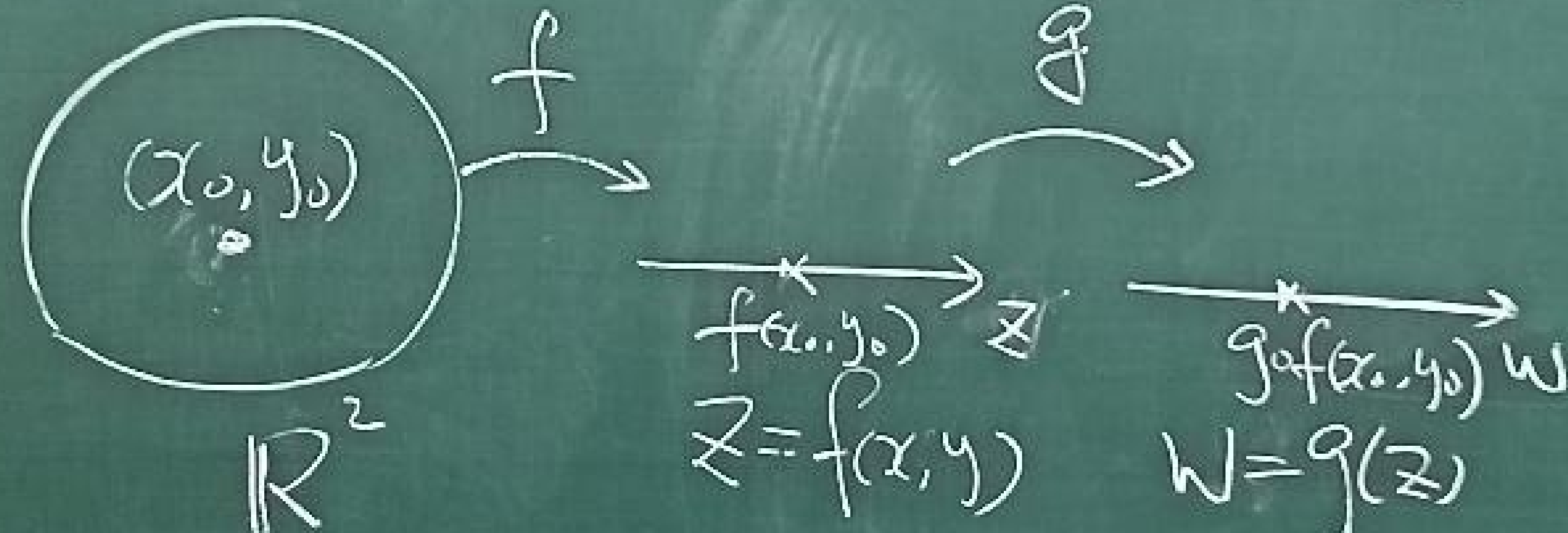
(Lecture 12, page 2-3)

<u>Def</u>: f is continuous in a region R if f is continuous at every $(x,y)\in R$

Remark. If

$$\begin{cases} f \text{ is cont. at } (x_0, y_0) \\ g \text{ is cont. at } z_0 = f(x_0, y_0) \end{cases}$$

Then $g \circ f$ is cont. at (x_0, y_0)



$$g \circ f(x, y) = g(f(x, y))$$

Eg $\cos\left(\frac{xy}{x^2+1}\right)$

$$(x, y) \xrightarrow{f} z = \frac{xy}{x^2+1} \xrightarrow{g} w = \cos z$$

Partial Derivatives

Def $$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

Notations $$\frac{\partial f}{\partial x} = f_x = \partial_x f = \partial_1 f$$

$$\frac{\partial f}{\partial y} = f_y = \partial_y f = \partial_2 f$$

Eg1: $f(x, y) = x^2 + 3xy + y - 1$

$$\partial_x f = 2x + 3y + 0$$

$$\partial_y f = 0 + 3x + 1$$

Eg 2 $f(x,y,z) = x\sin(y+3z)$

$$\partial_x f \underset{\substack{y \sim \text{const} \\ z \sim \text{const}}}{=\!=} \sin(y+3z)$$

$$\partial_y f \underset{\substack{x \sim \text{const} \\ z \sim \text{const}}}{=\!=} x\cos(y+3z)$$

$$\partial_z f \underset{\substack{x \sim \text{const} \\ y \sim \text{const}}}{=\!=} x\underbrace{\cos(y+3z)}_{\sin'} \cdot \underbrace{3}_{\partial_z(y+3z)}$$

Eg 3 Find $\frac{\partial z}{\partial x}$ at $(0,1,1)$ if $z(x,y)$ is implicitly defined by

$$yz + \ln z = x + y.$$

Ans:

i.e. $y\,z(x,y) + \ln z(x,y) = x + y$

$$\overset{\partial_x}{\Longrightarrow} \; y z_x + \frac{1}{z} \cdot z_x = 1 + 0$$

$$\Longrightarrow \frac{\partial z}{\partial x}(x,y) = \frac{1}{y + \frac{1}{z}}$$

$\left(\underline{\text{Remark:}} \; (0,1,1) \text{ Satisfies } yz + \ln z = x + y \right)$

$$\underline{\text{Ans}} = \left. \frac{1}{y + \frac{1}{z}} \right|_{(0,1,1)} = \frac{1}{2}$$

Higher partial derivatives

$$z = f(x, y)$$

$$\frac{\partial^2 f}{\partial y^2} = \partial_y^2 f = \partial_y(\partial_y f) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x^2} = \partial_x^2 f = \partial_x(\partial_x f) = f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \partial_x(\partial_y f) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \partial_y(\partial_x f) = f_{xy}$$

Eg: $f(x, y) = x^2 + y^2$

$$\partial_x f = 2x, \quad \partial_y f = 2y$$

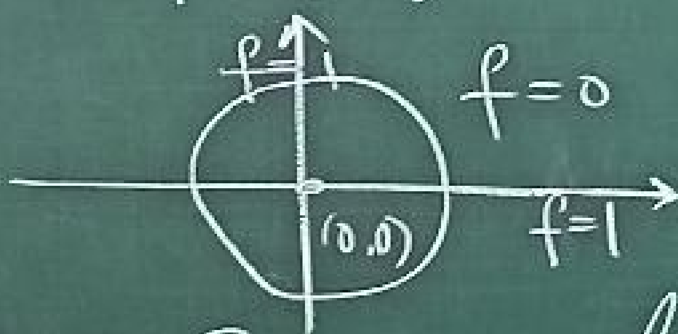
$$\partial_x^2 f = \partial_x(2x) = 2, \quad \partial_y^2 f = \partial_y(2y) = 2$$

$$\partial_x \partial_y f = \partial_x(2y) = 0, \quad \partial_y \partial_x f = \partial_y(2x) = 0$$

Eg 4 $\left.\begin{array}{l}\partial_x f(x_0,y_0)\\ \partial_y f(x_0,y_0)\end{array}\right\}$ exist $\not\Rightarrow$ f is cont. at (x_0,y_0)

$$f(x,y)=\begin{cases}0 & xy\neq 0\\ 1 & xy=0\end{cases}$$

(i) $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist



(ii) $$\partial_x f(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}$$

$$=\lim_{x\to 0}\frac{1-1}{x-0}=0.$$

(iii) Similarly $\partial_y f(0,0)=0$

Def $z = f(x,y)$ is differentiable at (x_0, y_0) if

(i) $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist.

(ii) $f(x,y)$ $\quad (*)$

$$= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \quad (**)$$

$$+ \varepsilon_1 \cdot (x-x_0) + \varepsilon_2 \cdot (y-y_0)$$

$$\left(\text{or } + \varepsilon \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$$

where $\lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon_1, \varepsilon_2, \varepsilon = 0$

(i.e. $(*)$ and $(**)$ are tangent at $(x_0, y_0, f(x_0,y_0))$)

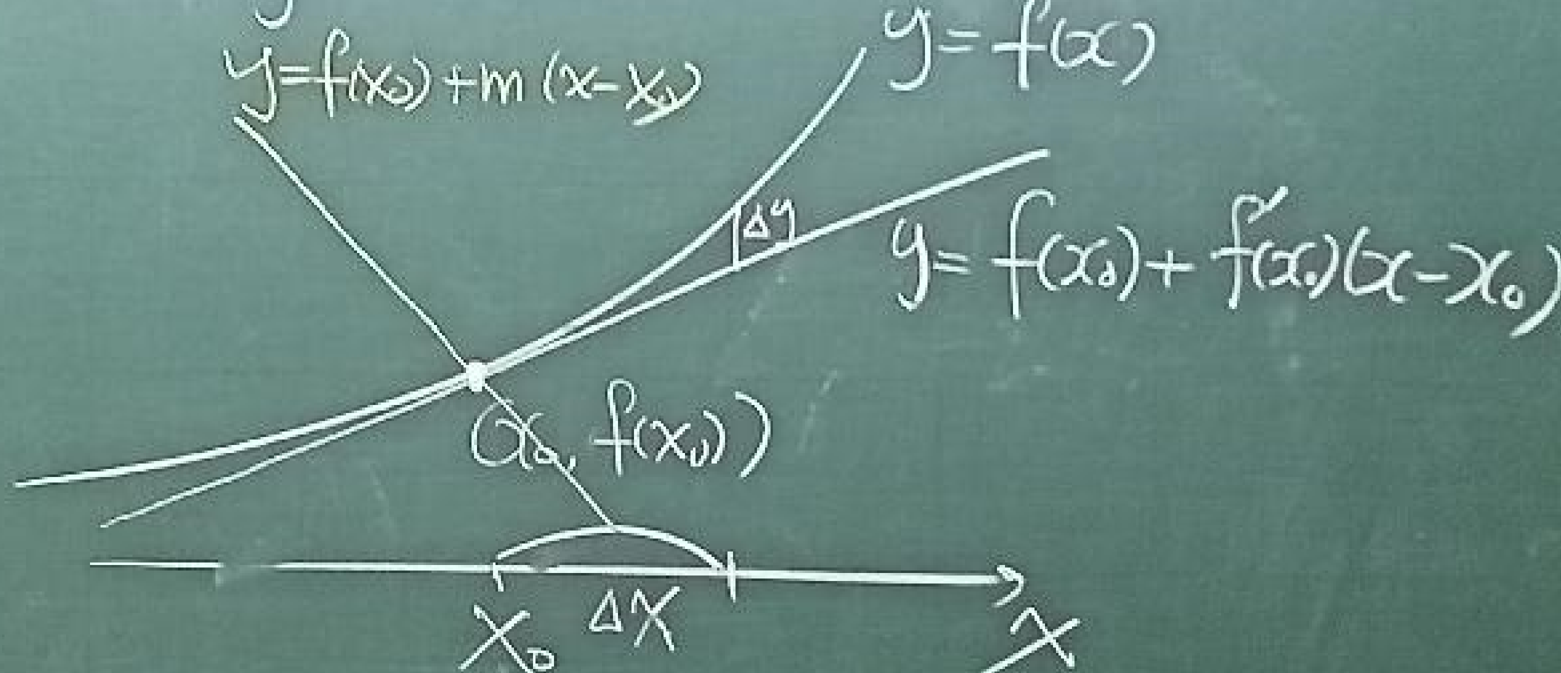
Re

an

W

t

Remark: differentiability and tangent line in 1D



$$\lim_{x\to x_0} \frac{f(x)-\left(f(x_0)+f'(x_0)(x-x_0)\right)}{x-x_0}=0$$

We may intepret the two curves $\begin{cases} y=f(x) \\ y=g(x) \end{cases}$ to be tangent at $(x_0, f(x_0))$

if $\displaystyle\lim_{x\to x_0} \frac{f(x)-g(x)}{x-x_0}=0$ ($|f(x)-g(x)|$ is much smaller than $|x-x_0|$)

Are the two curves

Eg5 $\begin{cases} y = e^x \\ y = 1-x \end{cases}$

tangent at $(0,1)$?

Ans $\displaystyle\lim_{x\to 0} \frac{e^x-(1-x)}{x-0} \overset{?}{=} 0$

Ans $=2$. Not tangent

Eg6 $\begin{cases} y = e^x \\ y = 1+x \end{cases}$

are tangent at $(0,1)$

since $\displaystyle\lim_{x\to 0} \frac{e^x-(1+x)}{x-0} = 0$