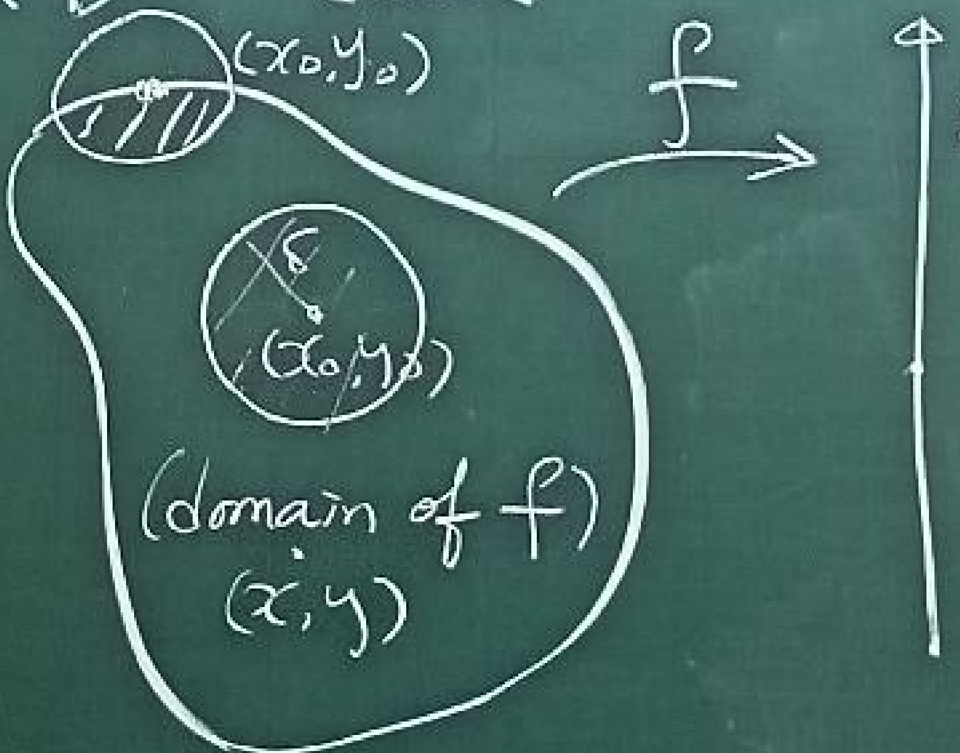
Limit and Continuity in higher dimension

2D case



$$f : (\text{domain of } f) \longmapsto \mathbb{R}$$

$$(x, y) \longmapsto z = f(x, y)$$

Def $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

if for every $\varepsilon > 0$,

there exists a corresponding

$\delta > 0$. such that for

all (x,y) in the domain of f

(*) $|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$(\Longleftarrow)$

(i.e. " $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ " $\Longrightarrow |f(x,y) - L| < \varepsilon$)

<u>Rm</u>: If $(*)$ is modified to

$(**)$ $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y)-L| < \varepsilon$

Then $(**)$ implies $(*)$,

in addition, $(**)$ requires

$(x-x_0)^2+(y-y_0)^2 = 0 \Rightarrow |f(x,y)-L| < \varepsilon$

That is, $|f(x_0,y_0)-L| < \varepsilon$

Since $\varepsilon>0$ is arbitrary

This means $f(x_0,y_0)-L=0$

i.e. $f(x_0,y_0) = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$ (i.e. f is continuous at (x_0,y_0))

Eg 1 $\lim_{(x,y)\to(x_0,y_0)} x = x_0$

Pf: for any $\varepsilon > 0$

find $\delta > 0$ such that

$|x - x_0| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$|f(x,y) - L|$

Ans: take $\delta = \varepsilon$

Then $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta = \varepsilon$

$\Rightarrow 0 < (x-x_0)^2 + (y-y_0)^2 < \varepsilon^2$

$\Rightarrow |x - x_0| < \varepsilon$

Eg. $\lim\limits_{(x,y)\to(0,1)} \dfrac{x-xy+3}{x+y^2}$

Ans $= \dfrac{0-0\cdot 1+3}{0+1^2} = 3$

Here f_1 and f_2 are just two random names for two different functions

The underlying reason is

$f_1(x,y)=x$ is continuous (Eg 1), similarly, $f_2(x,y)=y$

and their sum, product, diff, quotient are all continuous

$\therefore \lim\limits_{(x,y)\to(0,1)} =$ replace (x,y) by $(0,1)$

Eg3 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)\cdot x}{\sqrt{x}-\sqrt{y}}$

$= \lim\limits_{(x,y)\to(0,0)} (\sqrt{x}+\sqrt{y})\cdot x$

$= 0$

<u>Rm</u> domain of f

$= \left\{ \begin{array}{l} x\geq 0,\ y\geq 0 \\ x\neq y \end{array} \right\}$

Eg 4 $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (\text{"}r\to 0\text{"})}} \frac{4xy^2}{x^2+y^2}$

In polar coordinate

$$0<r<\delta \overset{?}{\Longrightarrow} |f(x,y)-L|<\varepsilon$$

$$(x=r\cos\theta,\ y=r\sin\theta)$$

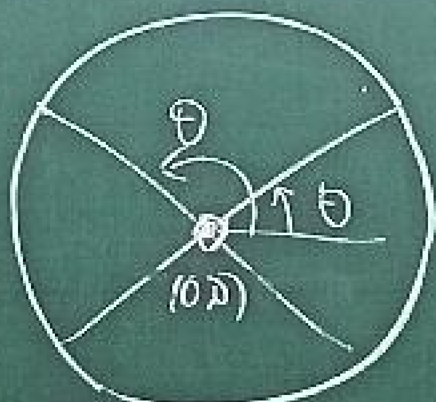
$$\left[\lim_{r\to 0}\frac{4xy^2}{x^2+y^2}=\lim_{r\to 0}\frac{4r^3\cos\theta\sin^2\theta}{r^2}=0\right]$$

Ans: $=0$.

For any $\varepsilon>0$. take $\delta=\frac{\varepsilon}{4}$

$$0<\underset{r}{\underset{\parallel}{\sqrt{\ }}}<\delta \Rightarrow |f(x,y)-0| = |4r\cos\theta\sin^3\theta| \leq 4r<4\delta=\varepsilon$$

Eg5 $\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{x}$ exists?



$0<\sqrt{x^2+y^2}<\delta$

In polar coord.

$$\frac{y}{x}=\frac{r\sin\theta}{r\cos\theta}=\tan\theta$$

Varies on different θ

Two path theorem: let $(x,y)\to(0,0)$ along two different path.

Here we take the paths: $y=mx$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}}\frac{y}{x}=\lim_{x\to0}\frac{mx}{x}=m$$

(different m give different limit
$\therefore$ limit does not exist)

Two Path Theorem

If $f(x,y)$ have different limits along two paths passing through (x_0, y_0) as $(x,y) \longrightarrow (x_0, y_0)$

Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Eg6 $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$

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See Figure 14.14, page 819

<u>Ans</u> let $(x,y)\to(0,0)$

along $y=mx$, $m\in\mathbb{R}$

$$\Rightarrow \lim \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2m}{1+m^2}$$

different $m \Rightarrow$ different lim

$\therefore \lim\limits_{(x,y)\to(0,0)}$ does not exist (Two Path Theorem)

Eg 7 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = ?$

See Figure 14.15, page 820

Sol Let $y = mx$, $x \to 0$

$$\lim_{x\to 0} \frac{2x^2 mx}{x^4+m^2x^2} = \begin{cases} 0 & m=0 \\ 0 & m\neq 0 \end{cases}$$

However, if we let $\dfrac{y=kx^2}{x\to 0}$

$$\lim_{x\to 0} \frac{2kx^4}{x^4(1+k^2)} = \frac{2k}{1+k^2}$$

$\therefore$ Two Path Thm $\Rightarrow$ limit does not exist