Known Taylor Series
(Memorize, forward and backward)

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \cdots \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad x \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad x \in \mathbb{R}$$

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \cdots \quad |x| < 1$$

Also for x=1

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad |x| < 1$$

Also for x=+-1

Eg1. $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

$$= \lim_{x\to 0} \frac{x - \sin x}{x^2 \sin x}$$

$$= \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3!} + \cdots\right)}{x^2\left(x - \frac{x^3}{3!} + \cdots\right)}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3!} + \cdots}{x^3 + \cdots}$$

$$= \frac{1}{6}$$

Eg2 $\frac{1}{1\cdot 2^1} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots = ?$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \Big|_{x=\frac{1}{2}}$$

$= ?$

Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ (What we want is f(1/2))

ratio test $\Rightarrow R = 1$

$\Rightarrow$ Term by term diff, integ, Multiplication ...... are all valid on $|x|<1$

$$\Rightarrow f'(x) = 1 - x + x^2 - \cdots = \frac{1}{1+x}$$

$$f(\tfrac{1}{2}) - \underset{0}{\underset{\parallel}{f(0)}} = \int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx = \ln(1+\tfrac{1}{2})$$

Eg (i) $x+2x^2+3x^3+\cdots=?$; $|x|<1$

(ii) $x+2^2x^2+3^2x^3+\cdots=?$

Ans: (i) $=\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1}$

$$= x\left(\sum_{n=1}^{\infty} x^n\right)'$$

$$= x\left(\sum_{n=0} \right)'$$

$$= x\left(\frac{1}{1-x}\right)' = \cdots$$

(ii) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} nx^{n-1}$

$\left(\frac{1}{1-x}\right)'' = \sum_{n=2} n(n-1)x^{n-2}$

See Lecture 07, page 7 for the remaining details

Remark. Formally, we can write for $\theta \in \mathbb{R}$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots$$

$$\stackrel{\text{def}}{=} \cos\theta + i\sin\theta$$

All algebraic manipulations, differentiation, integration on $\cos\theta + i\sin\theta$ can be applied to $e^{i\theta}$ correctly and more easily

Eg: $\int e^{ax} \cos bx \, dx$

Method 1: integration by part twice.

Method 2: $\cos bx = \mathrm{Re}(e^{ibx})$

$$\text{Ans} = \mathrm{Re}\left(\int e^{ax} \cdot e^{ibx} \, dx\right)$$

$$= \mathrm{Re}\left(\int e^{(a+bi)x} \, dx\right)$$

$$= \mathrm{Re}\left(\frac{e^{(a+bi)x}}{a+bi} + C\right)$$

$$= \mathrm{Re}\left(\frac{e^{ax}(\cos bx + i\sin bx)}{a+bi} \cdot \frac{a-bi}{a-bi} + C \right)$$

$$= \frac{e^{ax}(a\cos bx + a\sin bx)}{a^2+b^2} + C \quad (*)$$

Check $\frac{d}{dx}(*) = e^{ax}\cos bx$

$\therefore$ Correct.

Notation:

Re( ) = real part of a complex number.

If A, B are real, then Re(A+Bi) = A.