Binomial Series

$f(x) = (1+x)^m, \; m \in \mathbb{R}$

$T_{f,0}(x) = ?$

(1) If $m \in \mathbb{N}$

$f(x) =$ Polynomial.

$\Rightarrow T_{f,0}(x) = f(x)$

(2) If $m \notin \mathbb{N}$

$$f(x) = (1+x)^{m}$$

$$f'(x) = m(1+x)^{m-1}$$

$$f''(x) = m(m-1)(1+x)^{m-2}$$

$$\cdots\cdots$$

$$\Rightarrow f^{(k)}(0) = m(m-1)\cdots(m-k+1)$$

$$\therefore f(x) = P_n(x) + R_n(x)$$

where $P_n(x) = \sum_{k=0}^{n} \binom{m}{k} x^k$

$$R_n(x) = \binom{m}{n+1}(1+c_{n+1})^{m-n-1} x^{n+1}$$

$\therefore T_{f,0}(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k$

(i) Does $T_{f,0}(x)$ converge?

$$\binom{m}{k} = \frac{m(m-1)\cdots\cdots(m-k+1)}{k!}$$

Ratio test $\Rightarrow \rho = |x|$
(m fixed, $k \to \infty$)

$\therefore T_{f,0}(x)$ converges on $|x|<1$
div. on $|x|>1$

(ii) Does $T_{f,0}(x)$ converge to $f(x)$ on $|x|<1$?

$\lim_{n\to\infty} R_n(x) = 0$?

Not clear if $-1<x<0$

It can be shown indirectly (and not easily) that

$$\lim_{n\to\infty} R_n(x)=0 \text{ on } |x|<1$$

(Section ~~10.7~~ 10.10, problem 58)

$$\boxed{(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k}$$

for all $m \in \mathbb{R}$, $|x|<1$

Important! Memorize!

Eg1 $T_{\sin^{-1},0}(x) = ?$

Sol $(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}}, \ |x|<1$

$\sin^{-1}x - \sin^{-1}0 = \int_0^x (\sin^{-1}t)'\,dt$

(Fundamental Theorem of Calculus)

$= \int_0^x (1-t^2)^{-\frac{1}{2}}\,dt$ (Binomial, $m = \frac{-1}{2}$)

$= \int_0^x \Big(1 \underbrace{- \frac{1}{2}(-t^2)}_{= \frac{t^2}{2}} + \underbrace{\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}(-t^2)^2}_{= \frac{3}{8}t^4} + \cdots\Big)\,dt$

term by term integration

$= x + \dfrac{x^3}{6} + \dfrac{3}{40}x^5 + \cdots, \ |x|<1$

Thm A (Lecture 08, page 11) $\Longrightarrow\ = T_{\sin^{-1},0}(x)$ on $|x|<1$

Eg2. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

on $|x|<1$.

$\overset{x=1}{\overset{?}{\Longrightarrow}} \quad \frac{\pi}{4} \overset{?}{=} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Sol. $\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^n t^{2n}$

$+ \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}, \quad \underline{\forall t \in \mathbb{R}}$

$\left(1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}, \; x = -t^2 \right)$

$\Rightarrow \tan^{-1} x - \tan^{-1} 0 = \int_0^x \left(\frac{1}{1+t^2} \right) dt$

$= \int_0^x \left(1 - t^2 + t^4 - \cdots + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} \right) dt$

$= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \tilde{R}_n(x)$

Here $\tilde{R}_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} \, dt$

If $x = 1$

$$|\tilde{R}_n(1)| \leq \int_0^1 \left| \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} \right| dt$$

$$\leq \int_0^1 \frac{t^{2n+2}}{1} dt = \frac{1}{2n+3}$$

$$\Rightarrow \lim_{n \to \infty} \tilde{R}_n(1) = 0$$

(similarly for x = -1)

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

(Leibniz formula)

Leibniz's

Remark.

If $|t|<1$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$$

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If $|t|>1$

$$\frac{1}{1+t^2} = \frac{1}{t^2\left(1+\frac{1}{t^2}\right)}$$

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$$= \frac{1}{t^2}\left(1+\frac{1}{t^2}\right)^{-1} \quad (m=-1)$$

$$= \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} \cdots\right), \ |t|>1$$

Eg 3. $f(x) = \frac{1}{2-x}$, $|x|<2$

$T_{f,0}(x) = ?$ Ans: $= \frac{1}{2(1-\frac{x}{2})}$

$$= \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots\right)$$

Eg4 $\lim_{x\to 0} \dfrac{\sin x - \tan x}{x^3}$

Sol: Method 1: l'Hôpital Rule

Method 2:

$$= \frac{\sin x - \frac{\sin x}{\cos x}}{x^3}$$

$$= \frac{\frac{1}{2}\sin 2x - \sin x}{x^3 \cos x}$$

$$= \frac{\frac{1}{2}\left(\cancel{2x} - \frac{(2x)^3}{3!} + \cdots\right) - \left(\cancel{x} - \frac{x^3}{3!} + \cdots\right)}{x^3\left(1 - \frac{x^2}{2!} + \cdots\right)}$$

$$\lim_{x\to 0} = \frac{-4}{3!} + \frac{1}{3!} = \frac{-1}{2}$$