$$T_{f,a}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

For what values of x, does it converge?

i.e. $R = ?$

When it converges does it equal $f(x)$?

Eg $f(x) = \begin{cases} 0 & x=0 \\ e^{-1/x^2} & x \neq 0 \end{cases}$

$T_{f,0}(x) = 0 \neq f(x)$, for $x \neq 0$

Thm: (Taylor's Theorem)

If $f, f', f'', \ldots$

all exist on $|x-a|<\delta$

Then, for any $n \in \mathbb{N}$

$$f(x) = P_n(x) + R_n(x), \cdots (*)$$

on $|x-a|<\delta$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

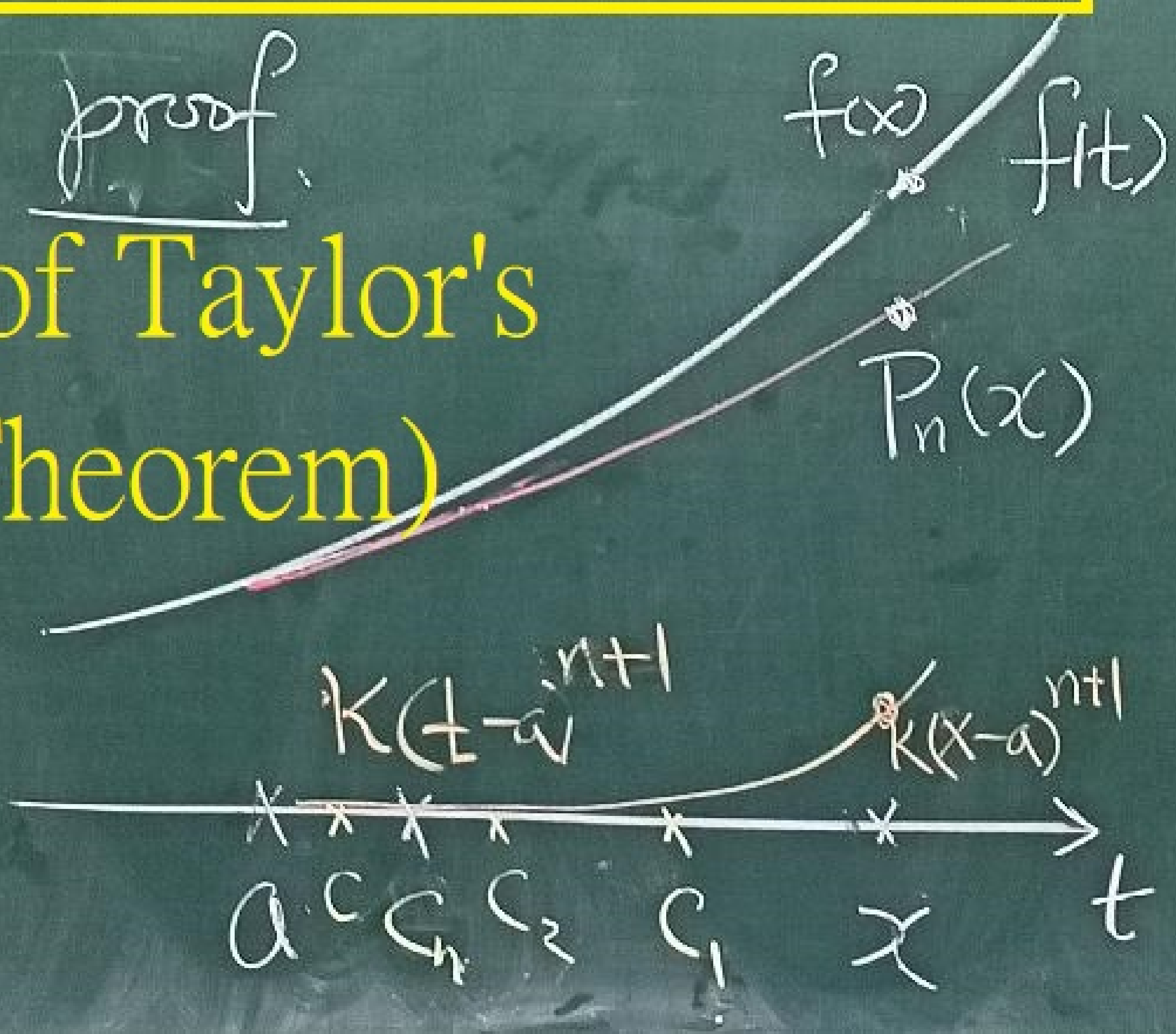
for some c between a and x

Since $T_{f,a}(x) = \lim_{n\to\infty} P_n(x)$

$$\boxed{\therefore\ T_{f,a}(x) = f(x) \iff \lim_{n\to\infty} R_n(x) = 0}$$

<u>proof</u>

(of Taylor's Theorem)



For any fixed x in $|x-a| < \delta$

We write

$$f(x) = P_n(x) + K(x-a)^{n+1}$$ only holds for this x

and find K (for this x)

(We want $K = \dfrac{f^{(n+1)}(c)}{(n+1)!}$)

For $|t-a| < \delta$, let

$$F(t) \stackrel{\text{def}}{=} f(t) - \left(P_n(t) + K(t-a)^{n+1}\right)$$ for all t

$$F(a) = f(a) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{(k!)}(a-a)^k = 0$$

$$F(x) = 0 \overset{\text{M.V.T.}}{\Longrightarrow} F'(c_1) = 0$$ for some c_1 between a and x

M.V.T. (Mean Value Thm)

$\therefore F'(c_1) = 0$

$$F'(a) = f'(a) - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^{k-1}\Big|_{t=a} - K(n+1)(t-a)^{n}\Big|_{t=a} = 0$$

$\overset{\text{M.V.T.}}{\Longrightarrow} F''(c_2) = 0$, some c_2 between a and c_1

(Mean Value Thm)

$\vdots$

$\Longrightarrow F^{(n+1)}(c) = 0$ for some c between a and c_n

(thus between a and x)

$$\| $$

$$f^{(n+1)}(c) - (0 + (n+1)!\,K)$$

$$\Longrightarrow K = \frac{f^{(n+1)}(c)}{(n+1)!}$$

This proves $(*)$ (on page 2)

$$T_{f,a}(x) = f(x)$$

$$\Longleftrightarrow \lim_{n\to\infty} \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} = 0$$

(Note: c also depends on n)

<u>Corollary</u>: If $|f^{(n+1)}(c)| \leq M$

for all c between a and x
and all $n \in \mathbb{N}$

then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \xrightarrow[n\to\infty]{} 0$

hence $T_{f,a}(x) = f(x)$

Eg 1. $f(x)=e^x$

$$T_{f,a}(x)=\sum_{k=0}^{\infty}\frac{e^a}{k!}(x-a)^k$$

$\xrightarrow{\text{Ratio}}$ $\rho=\lim_{k\to\infty}\frac{|x-a|}{k+1}=0$

$R=\infty$. $T_{f,a}(x)$ converges for any $x\in\mathbb{R}$

Moreover $R_n(x)=\frac{e^c}{(n+1)!}(x-a)^{n+1}$

$e^c\le\max(e^a,e^x)$

$\therefore \lim_{n\to\infty}R_n(x)=0$ $\therefore T_{f,a}(x)=e^x$ for any $x\in\mathbb{R}$

Eg 2 $f(x) = \sin x$, $a = 0$

$$T_{\sin,0}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\sin x = P_n(x) + R_n(x)$$

assume $n = 2l+1$ odd

$$\Rightarrow P_n(x) = \sum_{k=0}^{l} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^l x^{2l+1}}{(2l+1)!}$$

Firstly: Ratio test $\Rightarrow R = \infty$

Secondly: $R_n(x) = \dfrac{\frac{d^{n+1}}{dx^{n+1}} \sin x \Big|_{x=c}}{(n+1)!} x^{n+1}$

$$\Rightarrow |R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \xrightarrow[n\to\infty]{} 0 \Longrightarrow T_{\sin,0}(x) = \sin x \text{ for any } x \in \mathbb{R}$$

Application: Find approximate value of $\int_0^{\frac{1}{2}} \sin(t^2)\,dt$ and estimate the error.

Ans: $\int_0^{\frac{1}{2}} \sin t^2\,dt$

$= \int_0^{\frac{1}{2}} \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \cdots\right) dt$

term by term integration

$= \frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} \cdots \Big|_0^{\frac{1}{2}}$

$= \frac{1}{3\cdot 2^3} - \frac{1}{7\cdot 3!\, 2^7} + \frac{1}{11\cdot 5!\, 2^{11}} - \cdots$

$$\therefore \int_0^{\frac{1}{2}} \sin(t^2)\,dt$$

(term by term integration)

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^{\frac{1}{2}} (t^2)^{2k+1}\,dt$$

$$= \sum_{k=0}^{\infty} \underbrace{\boxed{\frac{(-1)^k \left(\frac{1}{2}\right)^{4k+3}}{(2k+1)!\,(4k+3)}}}_{=\,a_k}$$

= Alternating Series.

$$= \sum_{k=0}^{n} + E_n$$

$$|E_n| \le \left| a_{n+1} \right| = \frac{\left(\frac{1}{2}\right)^{4n+7}}{(2n+3)!\,(4n+7)}$$

For example, take n=2

$$\text{Approximation} = \sum_{k=0}^{2}$$

$$= \frac{1}{3\cdot 2^3} - \frac{1}{7\cdot 3!\,2^7} + \frac{1}{11\cdot 5!\,2^{11}}$$

$$|E_2| \leq \frac{2^{-15}}{7!\;15} \approx 4.04*10^{-10}$$

$$\cancel{\left(\leq \frac{1}{2}\cdot 10^{-9} \,?\right)}$$