Taylor Series

Question: For a given function $f(x)$ and $a \in \mathbb{R}$, can we always find $a_k \in \mathbb{R}$ and $R > 0$ such that

$$(*) \quad f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on $|x-a| < R$?

<u>Ans</u>: Not necessarily.

(See example in page 12)

But if $a_k \in \mathbb{R}$ and $R > 0$

do exist, we must

have

$$a_n = \frac{f^{(n)}(a)}{n!}$$

from term by term

differentiation Theorem.

$$\therefore f^{(n)}(a) = n!\, a_n$$

Question:

If $f^{(n)}(a)$ exist for all $n = 0, 1, 2, \cdots$

Is it necessarily true that $(*)$ holds for some $R > 0$?

Ans: Not necessarily. (Will explain later)

Also see example in page 12

Def: The Taylor Series generated by f at $x=a$ is $T_{f,a}(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$

(= Mclaurin Series if $a=0$)

Def: The Taylor Polynomial of degree n generated by f at $x=a$:

$$P_{n,a}(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Eg 1 (f is a polynomial)

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_5 x^5$$

Find $P_{3,0}(x)$, $P_{5,0}(x)$, $P_{7,0}(x)$, $T_{f,0}(x)$

<u>Ans</u>: $f^{(k)}(0) = k!\, a_k$, $0 \le k \le 5$

$$\therefore P_{3,0}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$f^{(l)}(0) = 0 \text{ for } l > 5$$

$$\Rightarrow P_{5,0}(x) = P_{7,0}(x) = T_{f,0}(x) = f(x)$$

<u>Remark</u>. If $f(x)$ is a polynomial of degree n then

$$P_{n,a}(x) = T_{f,a}(x) = f(x)$$

for any $a \in \mathbb{R}$.

- $f(x) = a_0 + a_1 x + \cdots + a_n x^n = P_{n,0}(x)$

$$= b_0 + b_1(x-1) + \cdots + b_n(x-1)^n = P_{n,1}(x)$$

$$\left(b_k = \frac{f^{(k)}(1)}{k!} \right)$$

$$\therefore P_{n,1}(x) = f(x) = T_{f,1}(x)$$

Eg2: $f(x) = e^x$, $T_{f,a}(x) = ?$

<u>Ans</u>: $\left.\dfrac{d^k f(x)}{dx^k}\right|_{x=a} = \left. e^x \right|_{x=a} = e^a$

$$T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{e^a}{k!}(x-a)^k$$

$$= e^a \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}$$

<u>Rm</u> from ratio test, $\rho = 0$ for any $x \in \mathbb{R}$. $\Rightarrow R = \infty$

$\therefore T_{f,a}(x)$ converges for any $x \in \mathbb{R}$

$(= e^x?$ <u>Ans</u>: Yes. (next time))

Eg 3. $T_{\cos(x),0}(x) = ?$

<u>Sol</u> $\cos^{(n)}(0) = ?$

$n=0$ $4, 8, \ldots$	$n=1$ $5, 9, \ldots$	$n=2$ $6, 10, \ldots$	$n=3$ $7, 11, \ldots$
$\cos(0)$	$-\sin(0)$	$-\cos(0)$	$\sin(0)$
$\parallel$	$\parallel$	$\parallel$	$\parallel$
1	0	-1	0

$$\Rightarrow T_{\cos x,0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad (背!)$$

Eg 4 $T_{\sin x, 0}(x) = ?$

Ans: $\overset{(\#)}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{x^{2\ell+1}}{(2\ell+1)}$$

(還有!)

HW: fill in the details for (#).

Eg 5 $f(x)=\frac{1}{x}$, $T_{f,2}(x)=?$

Sol. $f(x)=x^{-1}$, $f'(x)=-x^{-2}$

$f''(x)=2x^{-3}$, $f'''(x)=-6x^{-4}$

$$\Rightarrow f^{(k)}(x)=(-1)^k k!\, x^{-k-1}$$

$$\Rightarrow T_{f,2}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k k!}{k!}2^{-k-1}(x-2)^k$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{-(x-2)}{2}\right)^k$$

done!

Remark: $=\frac{1}{2}\,\frac{1}{1+\left(\frac{x-2}{2}\right)}$ $\left(\left|\frac{x-2}{2}\right|<1\right)$

$=\frac{1}{x}$ (= f(x) on $\left|\frac{x-2}{2}\right|<1$)

If f(x) already has a power series representation, then the Taylor series must exist and equals it.

Rm If $f(x)=\sum_{k=0}^{\infty} a_k(x-a)^k$ on $|x-a|<R$, $R>0$

$\Rightarrow \underline{f^{(n)}(a)=n!\,a_n}$, $n\in\mathbb{N}$ term by term differentiation

$\Rightarrow T_{f,a}(x)=\sum_{k=0}^{\infty}\frac{k!\,a_k}{k!}(x-a)^k$

$=f(x)$ (on $|x-a|<R$)

Eg5. Method 2:

$$\frac{1}{x}=\frac{1}{(x-2)+2}=\frac{1}{2}\left(\frac{1}{1+\frac{x-2}{2}}\right)$$

$\underline{\underline{\text{Geometric Series}}}$ $\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{-(x-2)}{2}\right)^k \underset{\text{from Rm above}}{=} T_{f,2}(x)$

In this example, all derivatives of f(x) at 0 are zero, hence the Taylor series is the zero function, different from f(x) since f(x) > 0 if x ≠ 0.

<u>Eg6</u> $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$T_{f,0}(x) = ?$ $\left(\begin{array}{l} f^{(n)}(0) = 0 \ (hw) \\ \Rightarrow T_{f,0}(x) = 0 \end{array} \right)$

<u>Ans</u>: $f(0) = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{e^{-\frac{1}{h^2}}}{h} \quad \left(\text{“}\frac{0}{0}\text{”} \right)$$

Applying l'Hopital's rule here will not give you the answer

Learn this trick

$$= \lim_{h \to 0} \frac{\left(\frac{1}{h}\right)}{e^{\frac{1}{h^2}}} \quad \left(\text{“}\frac{\infty}{\infty}\text{”} \right)$$

$$\overset{\text{l'Hôpital}}{=} \lim_{h \to 0} \frac{-h^{-2}}{-2h^{-3} e^{\frac{1}{h^2}}} = \lim_{h \to 0} \frac{h}{2e^{\left(\frac{1}{h^2}\right)}}$$

$= 0$ (Hw: $f''(0) = 0, \ldots, f^{(n)}(0) = 0$)

Remark on Lecture 08

Regarding the example in page 12:

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases} \tag{1}$$

It is known that

$$f^{(k)}(0) = 0 \text{ for all } k = 0, 1, 2, \cdots . \tag{2}$$

Take (2) for granted for now. See also homework 04 for verification of (2).

Recall the definition

$$T_{f,0}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k$$

It follows from (2) that

$$T_{f,0}(x) \equiv 0$$

That is, the Taylor series generated by f centered at $x = 0$ is the zero function. Therefore

$$f(x) \neq T_{f,0}(x) \text{ for any } x \neq 0 \tag{3}$$

since $f(x) > 0$ for $x \neq 0$.

This example gives a counter example for Question 1 (page 1) and Question 2 (page 3):

Question 2 (page 3):

Is it always true that $f(x) = T_{f,0}(x)$ on $|x - 0| < R$ for some $R > 0$?

Answer:

No. Take $f(x)$ as in (1) and see (3) above.

Question 1 (page 1):

For any given function $f(x)$ and $a \in \mathbb{R}$, can we always find $a_k \in \mathbb{R}$ and $R > 0$, such that

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k \text{ on } |x - a| < R? \tag{4}$$

Answer:

No. Take $f(x)$ as in (1) and $a = 0$. From Term by Term Differentiation Theorem, the only candidate for a_k is

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

However, this a_k does not satisfy (4) in view of (3) above.