

Taylor Series

Question: For a given

function $f(x)$, ^{and} $a \in \mathbb{R}$, can we

^{always} find $a_k \in \mathbb{R}$ and $R > 0$
such that

$$(*) \quad f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on $|x-a| < R$?

Ans: Not necessarily
(See example in page 12)

But if $a_k \in \mathbb{R}$ and $R > 0$
do exist, we must
have

$$a_n = \frac{f^{(n)}(a)}{n!}$$

from term by term
differentiation Theorem.

$$\therefore f^{(n)}(a) = n! a_n$$

Question

If $f^{(n)}(a)$ exist
for all $n=0,1,2,\dots$

Is it necessarily
true that (*)
holds for some
 $R > 0$?

Ans. Not necessarily.
(Will explain later)

Also see example in page 12

Def: The Taylor Series
Generated by f at $x=a$

is
$$T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

(= Maclaurin Series if $a=0$)

Def: The Taylor Polynomial
of degree n generated by f at
 $x=a$:

$$\underline{\underline{P_{n,a}(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k}}$$

Ex 1. (f is a polynomial)

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_5x^5$$

Find $P_{3,0}(x)$, $P_{5,0}(x)$, $P_{7,0}(x)$, $T_{f,0}(x)$

Ans: $f^{(k)}(0) = k! a_k, 0 \leq k \leq 5$

$$\therefore P_{3,0}(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$f^{(l)}(0) = 0 \text{ for } l > 5$$

$$\Rightarrow P_{5,0}(x) = P_{7,0}(x) = T_{f,0}(x) = f(x)$$

Remark. If $f(x)$ is a polynomial of degree n , then

$$P_{n,a}(x) = T_{f,a}(x) = f(x) \text{ for any } a \in \mathbb{R}.$$

$$\begin{aligned} f(x) &= a_0 + a_1x + \dots + a_nx^n = P_{n,0}(x) \\ &= b_0 + b_1(x-1) + \dots + b_n(x-1)^n = P_{n,1}(x) \\ &\quad \left(b_k = \frac{f^{(k)}(1)}{k!} \right). \end{aligned}$$

$$P_{n,1}(x) = f(x) = T_{f,1}(x)$$

Eg2: $f(x) = e^x$, $T_{f,a}(x) = ?$

Ans: $\frac{d^k f(x)}{dx^k} \Big|_{x=a} = e^x \Big|_{x=a} = e^a$

$$T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{e^a}{k!} (x-a)^k$$
$$= e^a \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} \quad \left(\frac{e^t}{t!} \right)$$

Rm from ratio test, $\rho = 0$
for any $x \in \mathbb{R} \Rightarrow R = \infty$

$T_{f,a}(x)$ converges for any $x \in \mathbb{R}$
($= e^x$?) Ans: Yes, (next time)

$$\text{Eg 3 } T_{\cos(x), 0}(x) = ?$$

$$\text{Sol } \cos^{(n)}(0) = ?$$

$n=0$ 4, 8, ...	$n=1$ 5, 9, ...	$n=2$ 6, 10, ...	$n=3$ 7, 11, ...
$\cos(0)$	$-\sin(0)$	$-\cos(0)$	$\sin(0)$
1	0	-1	0

$$\Rightarrow T_{\cos(x), 0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \left(\frac{d^k}{dx^k} \right)$$

Ex 4 $T_{\sin x, 0}(x) = ?$

Ans: (#) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$$
$$= \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l+1}}{(2l+1)!}$$

HW: fill in the details for (#).

$$\text{Ex 5 } f(x) = \frac{1}{x}, \quad T_{f,2}(x) = ?$$

$$\text{Sol: } f(x) = x^{-1}, \quad f'(x) = -x^{-2}$$

$$f''(x) = 2x^{-3}, \quad f'''(x) = -6x^{-4}$$

$$\Rightarrow f^{(k)}(x) = (-1)^k k! x^{-k-1}$$

$$\Rightarrow T_{f,2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{k!} 2^{-k-1} (x-2)^k$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{-(x-2)}{2} \right)^k$$

done!

$$\text{Remark: } = \frac{1}{2} \frac{1}{1 + \left(\frac{x-2}{2}\right)} \left(\left| \frac{x-2}{2} \right| < 1 \right)$$

$$= \frac{1}{x} \quad (= f(x) \text{ on } \left| \frac{x-2}{2} \right| < 1)$$

If $f(x)$ already has a power series representation, then the Taylor series must exist and equals it.

Rm If $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$

on $|x-a| < R, R > 0$

$\Rightarrow a_n = \frac{f^{(n)}(a)}{n!}, n \in \mathbb{N}$ term by term differentiation

$\Rightarrow T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{k! a_k}{k!} (x-a)^k$
 $= f(x) \text{ (on } |x-a| < R \text{)}$

Eg 5. Method 2.

$$\frac{1}{x} = \frac{1}{(x-2)+2} = \frac{1}{2} \left(\frac{1}{1 + \frac{x-2}{2}} \right)$$

Geometric Series

$$\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{-(x-2)}{2} \right)^k = T_{f,2}(x)$$

from Rm above

In this example, all derivatives of $f(x)$ at 0 are zero, hence the Taylor series is the zero function, different from $f(x)$ since $f(x) > 0$ if $x \neq 0$.

Eg 6 $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$T_{f,0}(x) = ?$ $\left(\begin{array}{l} f^{(n)}(0) = 0 \text{ (hw)} \\ \Rightarrow T_{f,0}(x) = 0 \end{array} \right)$

Ans: $f(0) = 0$. (1st A)

$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \rightarrow 0} \frac{e^{-\frac{1}{h^2}}}{h}$ ("0" over 0)

Applying l'Hopital's rule here will not give you the answer

Learn this trick $= \lim_{h \rightarrow 0} \frac{(\frac{1}{h})}{e^{\frac{1}{h^2}}}$ ("∞" over ∞)

l'Hopital $\lim_{h \rightarrow 0} \frac{-h^{-2}}{-2h^{-3} e^{\frac{1}{h^2}}} = \lim_{h \rightarrow 0} \frac{h}{2e^{\frac{1}{h^2}}} = 0$ (Hw: $f'(0) = 0, \dots, f^{(n)}(0) = 0$)

Remark on Lecture 08

Regarding the example in page 12:

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases} \quad (1)$$

It is known that

$$f^{(k)}(0) = 0 \text{ for all } k = 0, 1, 2, \dots \quad (2)$$

Take (2) for granted for now. See also homework 04 for verification of (2).

Recall the definition

$$T_{f,0}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (x-0)^k$$

It follows from (2) that

$$T_{f,0}(x) \equiv 0$$

That is, the Taylor series generated by f centered at $x = 0$ is the zero function. Therefore

$$f(x) \neq T_{f,0}(x) \text{ for any } x \neq 0 \quad (3)$$

since $f(x) > 0$ for $x \neq 0$.

This example gives a counter example for Question 1 (page 1) and Question 2 (page 3):

Question 2 (page 3):

Is it always true that $f(x) = T_{f,0}(x)$ on $|x-0| < R$ for some $R > 0$?

Answer:

No. Take $f(x)$ as in (1) and see (3) above.

Question 1 (page 1):

For any given function $f(x)$ and $a \in \mathbb{R}$, can we always find $a_k \in \mathbb{R}$ and $R > 0$, such that

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k \text{ on } |x-a| < R? \quad (4)$$

Answer:

No. Take $f(x)$ as in (1) and $a = 0$. From Term by Term Differentiation Theorem, the only candidate for a_k is

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

However, this a_k does not satisfy (4) in view of (3) above.