Algebraic Manipulations of power series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

and $B(x) = \sum_{n=0}^{\infty} b_n x^n$

both converge on $|x| < R$

Then $A(x) \pm B(x) = ?$

$A(x) \cdot B(x) = ?$

$A(x) / B(x) = ?$

Thm: If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$B(x) = \sum_{n=0}^{\infty} b_n x^n$ both

converge (abs) on $|x| < R$

and $C_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$ $\left(\sum_{k=0}^{n} a_k b_{n-k} \right)$

Then $\sum_{n=0}^{\infty} C_n x^n$ conv. abs.

on $|x| < R$ and $= A(x) \cdot B(x)$

Sol:

$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$

$A(x) \cdot B(x) = a_0 b_0 + (a_1 b_0 + a_0 b_1) x$
$+ (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$

How about power series of $\frac{A(x)}{B(x)}$? (if it exists)

Sol: If $C(x) = \frac{A(x)}{B(x)}$

$= \sum_{n=0}^{\infty} c_n x^n$ (converge on $|x| < r \le R$)

Then $A(x) = B(x) \cdot C(x)$ on $|x| < r$

$\Rightarrow a_0 = b_0 \cdot c_0 \quad (\Rightarrow c_0 = \frac{a_0}{b_0})$

$a_1 = b_0 c_1 + b_1 c_0$
$(\Rightarrow c_1 = (b_1 c_0 - a_1)/b_0)$

$a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0$

$\cdots\cdots$

Eg1 Take for granted that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

find first few terms of Power Series for $\frac{1}{e^x}$

Sol: Long division:

$$
\begin{array}{r|l}
 & 1 \quad -1 \quad +\tfrac{1}{2} \quad -\tfrac{1}{6} \\
\hline
1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \cdots & 1 + 0 + 0 + 0 + \cdots \\
 & 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \cdots \\
\cline{2-2}
 & \quad -1 - \tfrac{1}{2} - \tfrac{1}{6} + \cdots \\
 & \quad -1 - 1 - \tfrac{1}{2} + \cdots \\
\cline{2-2}
 & \qquad \tfrac{1}{2} \quad \tfrac{1}{3} \\
 & \qquad \tfrac{1}{2} \quad \tfrac{1}{2} \\
\cline{2-2}
 & \qquad\quad -\tfrac{1}{6}
\end{array}
$$

Ans:

$$= 1 - \frac{x}{1} + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

from result of long division

Term by term differentiation

Thm If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$

Converges (abs.) on $|x-a| < R > 0$

Then (1) $f', f'', \ldots, f^{(n)}, \ldots$

all exist on $|x-a| < R$

(2) $f'(x) = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$

$f''(x) = \sum_{n=2}^{\infty} n(n-1) C_n (x-a)^{n-2}$

$\vdots$

all converge (abs.) on $|x-a| < R$

Eg 2: $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$= \frac{1}{1-x}$ on $|x| < 1$

What is the power series representation of $f'(x) = \frac{1}{(1-x)^2}$?

<u>Ans</u>: $f'(x) = 1' + x' + (x^2)' + \cdots$

$= 1 + 2x + \cdots nx^{n-1} + \cdots$ on $|x| < 1$

<u>Rm</u> $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = ?$

<u>Ans</u>: $= \left(\sum_{n=1}^{\infty} n x^{n-1} \right) \Big|_{x=\frac{1}{2}}$

$= \left(\frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) \right) \Big|_{x=\frac{1}{2}} = \frac{1}{(1-x)^2} \Big|_{x=\frac{1}{2}} = 4$

Eg3: $\sum_{n=1}^{\infty} n^2 x^n = ?$ $|x|<1$

Sol: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\Rightarrow \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$n^2 x^n = x^2 (n(n-1) x^{n-2}) + x(n x^{n-1})$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^n = \underbrace{\sum_{n=1}^{\infty} n(n-1) x^n}_{=\sum_{n=2}^{\infty}} + \sum_{n=1}^{\infty} n x^n$$

$$= \frac{x^2 \cdot 2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x+x^2}{(1-x)^3}$$

Rm Term by term differentiation may not be valid for other kind of series

Eg: $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$

Sol: $|a_n| \le \frac{1}{n^2}$ $\therefore \sum_{n=1}^{\infty} a_n$ converges abs. on $x \in \mathbb{R}$

But $\displaystyle\sum_{n=1}^{\infty} n a_n = \sum_{n=1}^{\infty} \frac{n!}{n^2} \cos(n!x)$

diverges for any $x \in \mathbb{R}$.

Thm (term by term integration)

If $f(x)=\sum_{n=0}^{\infty} C_n(x-a)^n$

converges abs. on $|x-a|<R$

Then $\sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1}$

also converges abs. on $|x-a|<R$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1} + C$

$$\left[\int_a^x f(t)\,dt = \sum_{n=0}^{\infty}\int_a^x C_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\frac{C_n}{n+1}(x-a)^{n+1}\right]$$

Eg 3: Evaluate $F(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

on $|x|<1$. $\quad = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$

<u>Sol</u>: <u>Step 1</u>: Check radius of conv.

for F by root test $\Longrightarrow R=1$

ratio

<u>Step 2</u>.

$$F'(x) = 1 - x^2 + x^4 - x^6 + \cdots$$

$$= \frac{1}{1+x^2}$$

$$F(x) - \overbrace{F(0)}^{=0} = \int_0^x F'(t)\,dt$$

$$= \int_0^x \frac{1}{1+t^2}\,dt = \underline{\underline{\tan^{-1} x}}$$