Def: Power series

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

$$\left(\stackrel{\text{def}}{=} C_0 + \sum_{n=1}^{\infty} C_n (x-a)^n\right)$$

is a function of x.

a = center, C_n = coefficients

Question: For what values of x, is the power series convergent? (when C_n are given)

Eg1: $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$

is a Geometric Series

with $r = \frac{-(x-2)}{2}$

It converges $\Leftrightarrow |r| < 1$

In fact, it converges absolutely on $0 < x < 4$ and diverges elsewhere.

E.g.2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$(0! \stackrel{\text{def}}{=} 1)$

Ratio test:

$$|u_n| = \frac{|x|^n}{n!}$$

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = 0$$

for any $x \in \mathbb{R}$.

$\therefore$ It converges absolutely for all $x \in \mathbb{R}$

Eg 3 $\sum_{n=0}^{\infty} n!\,x^n$

Ratio test

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\begin{cases}0 & x=0\\ \infty, & x\neq 0\end{cases}$$

$$\left(=\lim_{n\to\infty}(n+1)|x|\right)$$

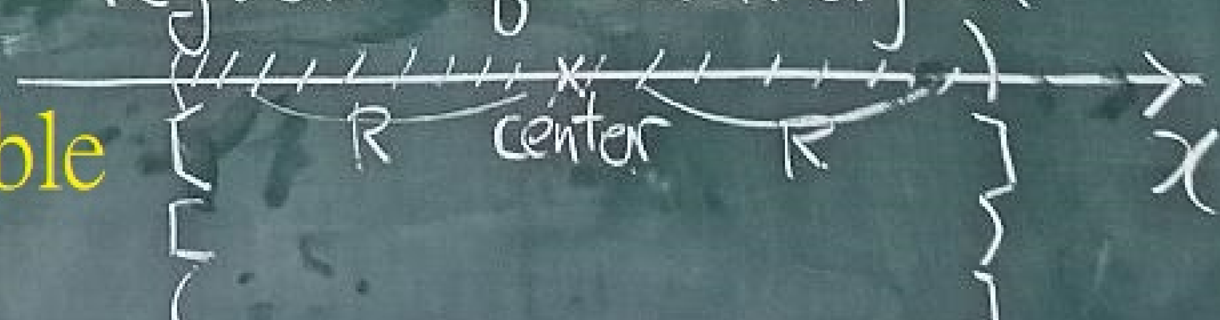
It converges at $x=0$ only, and diverges elsewhere.

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for $|x| < |c|$.

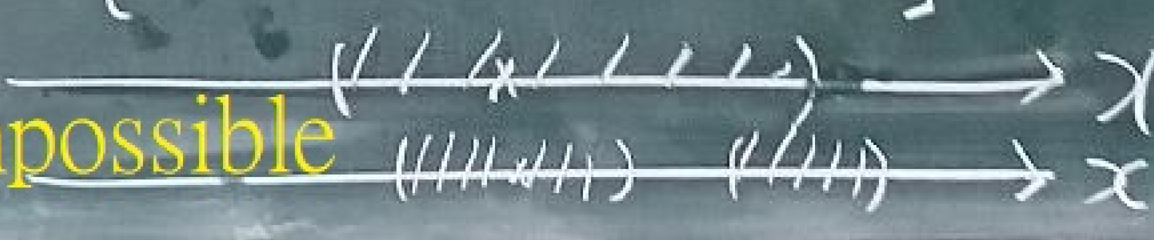
$\left(\Rightarrow \text{If it diverges at } x = d, \text{ then it diverges for } |x| > |d| \right)$

region of convergence

(O) possible



(X) (X) impossible



pf: If $\sum_{n=0}^{\infty} a_n c^n$ converges

$$\Rightarrow \lim_{n\to\infty} a_n c^n = 0$$

$$\Rightarrow |a_n c^n| < 1 \quad \forall n \geq N$$

There exists an integer N such that

$$\Rightarrow |a_n| < \frac{1}{|c|^n} \quad \forall n \geq N$$

If $|x| < |c|$

$$\sum_{n=N}^{\infty} |a_n x^n| < \sum_{n=N}^{\infty} \left(\frac{|x|}{|c|}\right)^n < \infty$$

i.e. conv abs. for $|x| < |c|$

(absolutely)

Remark: Possible cases for region of convergence.

(Center = 0)

(1) Converges absolutely for all $x \in \mathbb{R}$ (Radius of conv. $R = \infty$)

(2) Converges only at $x = 0$ (Radius of conv $R = 0$)

(3) $\exists\ 0 < R < \infty$

it $\begin{cases} \text{conv. abs. on } |x| < R \\ \text{diverges on } |x| > R \end{cases}$

$R \stackrel{\text{def}}{=}$ radius of conv. for $\sum a_n x^n$

In general, $0 \le R \le \infty$

In other words, the region of convergence must be an interval centered at the center called "interval of convergence"

Usually, the radius of convergence can be found by Ratio Test or Root test.

<u>Rem</u> $\sum_{n=0}^{\infty} a_n(x-a)^n$

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$ $\quad 0 \le \rho \le \infty$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n} \right| = \rho|x-a|$$

$$\Rightarrow \sum a_n(x-a)^n \begin{cases} \text{conv. abs. if } |x-a| < \frac{1}{\rho} \\ \text{div} \quad \text{if } |x-a| > \frac{1}{\rho} \end{cases}$$

$$\Rightarrow R = \frac{1}{\rho}$$

Similarly, if $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

Then $R = \frac{1}{\rho}$

Remark: R (radius of conv.) always exist for $\sum_{n=0}^{\infty} a_n (x-a)^n$

but $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ or $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$

may or may not exist.

Eg: $\sum_{n=1}^{\infty} a_n x^n$

$= \left(\frac{x}{2}\right)^1 + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + \cdots$

$\lim \frac{|a_{n+1}|}{|a_n|}$, $\lim |a_n|^{\frac{1}{n}}$ does not exist

Rm: $R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = 2$

reference only (beyond the scope of this course)

Eg 4 ⓐ $\sum_{n=0}^{\infty} x^n$ converges on $\underline{\underline{(-1,1)}}$

div. elsewhere.

ⓑ $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges $|x|<1$, $x=-1$

div on $|x|>1$, $x=1$

$a_n = \frac{1}{n} \xrightarrow[\text{Root}]{\text{Ratio}} \rho = 1$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 = \lim |a_n|^{\frac{1}{n}}$

$\begin{cases} x=1: \text{p-series}, p=1. \\ x=-1: \text{Alternating Series test} \end{cases}$

$\Rightarrow$ converges on $\underline{\underline{[-1,1)}}$

diverges elsewhere.

ⓒ $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $\rho=1$, $R=1$,

$x=\pm1$: conv. $\Rightarrow$ converges on $\underline{\underline{[-1,1]}}$

div elsewhere.