The Ratio Test

If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$

(1) $\sum_{n=1}^{\infty} |a_n| < \infty$ (*)

if $0 \le \rho < 1$

(2) $\sum_{n=1}^{\infty} a_n$ diverges

if $\rho > 1$

(*): $\sum_{n=1}^{\infty} a_n$ converges absolutely pf:

(1) If $0 \le \rho < 1$,

define $r = \frac{1+\rho}{2}$, $0<r<1$

Since $\lim_{n\to} \left|\frac{a_{n+1}}{a_n}\right| = \rho$

let $\varepsilon = \frac{1-\rho}{2} > 0$

$\Rightarrow \exists\ N \in \mathbb{N}$

such that

$$n > N \Rightarrow \left|\frac{|a_{n+1}|}{|a_n|} - \rho\right| < \varepsilon$$

$$\Rightarrow \frac{|a_{n+1}|}{|a_n|} < \rho + \varepsilon = \frac{1+\rho}{2} = r < 1$$

$$\therefore \sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N-1} + \sum_{n=N}^{\infty}$$

$$= \sum_{n=1}^{N-1}|a_n| + |a_N| + |a_{N+1}| + \cdots$$

$$\leq \sum_{n=1}^{N-1}|a_n| + |a_N| + r|a_N| + r^2|a_N| + \cdots\cdots$$

$$= \left(\sum_{n=1}^{N-1}|a_n|\right) + \text{Convergent Geometric Series}$$

$$< \infty$$

(2) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho > 1$, define $r = \frac{1+\rho}{2} > 1$

let $\varepsilon = \frac{\rho-1}{2} > 0$, $\Rightarrow \exists N \in \mathbb{N}$ such that

$n > N \Rightarrow \frac{|a_{n+1}|}{|a_n|} > \rho - \varepsilon = r > 1$ $\therefore \lim |a_n| \neq 0$

The proof for root test is similar.

Eg 1 ⓐ $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

Sol: Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}+5}{3\cdot(2^n+5)}$$

$$= \lim_{n\to\infty} \frac{2+5\cdot 2^{-n}}{3\cdot(1+5\cdot 2^{-n})}$$

$$= \frac{2}{3} < 1$$

$\Rightarrow$ convergent

Eg

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)}$$

$= 4$

$\therefore$ divergent (Ratio Test)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n\, n!\, n!}{(2n)!}$

Ratio Test $\Rightarrow \rho = 1$ (inconclusive)

But $\dfrac{a_{n+1}}{a_n} = \dfrac{4(n+1)(n+1)}{(2n+2)(2n+1)}$

$= \dfrac{(n+1)(n+1)}{(n+1)(n+\frac{1}{2})} > 1. \Rightarrow \lim a_n \neq 0$

$\therefore$ divergent

Eg 2 ⓐ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} \xrightarrow[\text{Root}]{\text{Ratio}} \rho = \frac{1}{2}$

$\therefore$ convergent.

ⓑ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3} \Rightarrow \rho = 2$, div.

ⓒ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n \xrightarrow{\text{(Root)}} \rho = 0$. conv.

ⓓ $a_n = \begin{cases} \dfrac{n}{2^n} & n \text{ is odd} \\[2mm] 1/2^n & n \text{ is even} \end{cases}$

$\dfrac{a_{n+1}}{a_n} = \begin{cases} \dfrac{1}{2n} & n \text{ is odd} \\[2mm] \dfrac{n+1}{2} & n \text{ is even} \end{cases}$

$\Rightarrow$ no conclusion by Ratio test.

$\neq 0$

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But Root test $\Rightarrow \rho = \frac{1}{2}$

Alternating Series Test
(Leibniz Test)

If (1) $u_n > 0$

(2) $u_n \geq u_{n+1}$

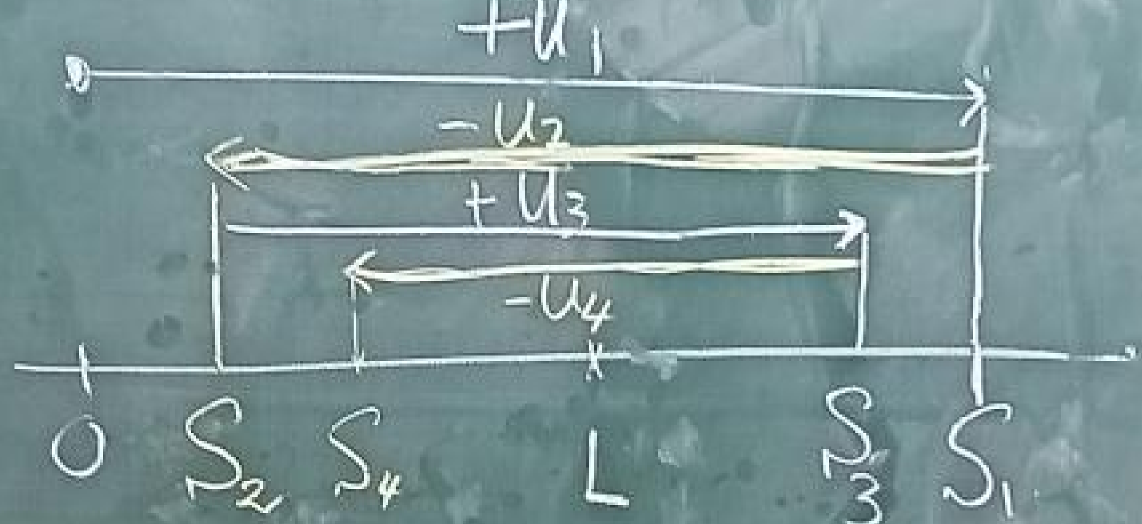
for all $n \geq N$.

(3) $\lim\limits_{n\to\infty} u_n = 0$

Then $\sum\limits_{n=1}^{\infty} (-1)^{n+1} u_n$ converges.

Eg: $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges

Pf: let $S_n = \sum_{k=1}^{n} (-1)^{k+1} u_k$



$$0 < S_2 < S_4 < \cdots < S_3 < S_1$$

$\therefore \{S_2, S_4, S_6, \cdots, S_{2k}, \cdots\}$

is an increasing sequence

and bounded above $(S_{2k} < S_1)$

From The Monotone Sequence Thm $\left(\begin{matrix}\text{Section 10.1} \\ \text{Thm 6}\end{matrix}\right)$

$\therefore \lim_{k\to\infty} S_{2k} = L$ for some $L \in \mathbb{R}$.

Moreover, $S_{2k+1} = S_{2k} + u_{2k+1}$

$$\xrightarrow[k\to\infty]{} L + 0.$$

$$\therefore \lim_{n\to\infty} S_n = L$$

<u>Remark</u>. (1) $S_{2k} < L < S_{2\ell-1}$

for any $k, \ell \in \mathbb{N}$

(2) $|S_n - L| < u_{n+1}$

$\because\ 0 < L - S_{2k} < u_{2k+1}$

$0 < S_{2\ell-1} - L < u_{2\ell}$

in (0,1)

n 6

Def $\sum a_n$ converge absolutely if $\sum |a_n| < \infty$

Def $\sum a_n$ converge conditionally

if $\begin{cases} \sum a_n \text{ converges} \\ \sum |a_n| = \infty \end{cases}$

Eg 3 (a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converge conditionally.

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$, $p>0$

Does it converge? absolutely or conditionally?

Sol: Step 1: it converges by Leibniz Test

Step2: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv.} & p>1 \\ \text{div.} & 0<p\le 1 \end{cases}$

(check absolute convergence)

Step3: $\displaystyle\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges (Leibnitz) $\underline{\underline{\text{abs.}}}$ $p>1$, $\underline{\underline{\text{cond.}}}$ $0<p\le 1$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$

Does it converge? absolutely or conditionally?

(i) converges (Leibniz Test) (= Alternating Series Test)

(ii) it converges cond. (conditionally)

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \text{ conv?}$$

(Ans: No)

Sol: Integral test:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$$

$$= \int_{x=2}^{\infty} \frac{1}{\sqrt{\ln x}}\,d(\ln x)$$

$$\left(\text{let } u = \ln x \quad \int_{u=\ln 2}^{\infty} \frac{1}{\sqrt{u}}\,du\right)$$

$$\text{or } \frac{dx}{x} = d(\ln x)$$

$$= \text{divergent } \left(p = \tfrac{1}{2}\right)$$