Recall

Thm 10 (The comparison Test)

If $0 \le d_n \le a_n \le c_n$

for all $n \ge N$

Then

① $\sum_{n=1}^{\infty} c_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

② $\sum_{n=1}^{\infty} d_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

Thm 11 (Limit Comparison Test)

If $a_n > 0$, $b_n > 0$ $\forall n \geq N$

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$

Then $\sum_{n=1}^{\infty} b_n < \infty \Leftrightarrow \sum_{n=1}^{\infty} a_n < \infty$

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Then $\sum_{n=1}^{\infty} b_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

Then $\sum_{n=1}^{\infty} b_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

a_n | b_n

Eg 1		Compare with	Thm	Result.
ⓐ	$\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n+1)^2}$	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$	11(i)	div.
ⓑ	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$	$\displaystyle\sum \frac{1}{2^n}$	11(i)	conv.
ⓒ	$\displaystyle\sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+5}$	$\displaystyle\sum \frac{\ln n}{n}$	11(i)	div.

$$\left(\text{Method 1: } \frac{\ln n}{n} > \frac{1}{n} \xrightarrow{10ⓑ} \sum \frac{\ln n}{n} = \infty\right)$$

or Method 2: $\displaystyle\int_3^{\infty} \frac{\ln x}{x}\,dx = \infty \Rightarrow \sum \frac{\ln n}{n} = \infty$

$\left(\dfrac{\ln x}{x} \text{ is decreasing for } x > e \cong 2.718\ldots\right)$

a_n $\quad$ b_n

Compare $\quad$ Thm $\quad$ Result

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ $\quad$ $\sum \frac{1}{n}$ $\quad$ 11(i) $\quad$ div.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ $\quad$ $\sum n^{\frac{-3}{2}}$ $\quad$ 11(i) $\quad$ conv.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{3}{2}}}$ $\quad$ $\sum \frac{n^{0.2}}{n^{\frac{3}{2}}}$ $\quad$ 11(ii) $\quad$ conv.

$$\frac{a_n}{b_n} = \left(\frac{\ln n}{n^{0.1}}\right)^2$$

$$\lim_{n\to\infty} \frac{\ln n}{n^{0.1}} \overset{\text{l'Hopital}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{0.1\, n^{-0.9}}$$

$$= \lim_{n\to\infty} \frac{10}{n^{0.1}} = 0$$

a_n $\quad b_n$

Compare Thm Res.

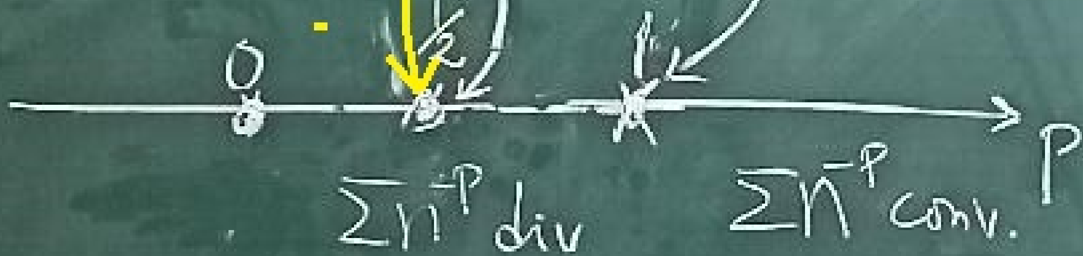
(g) $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{1}{n^2}\right)$ $\quad \displaystyle\sum \frac{1}{n^2}$ $\quad$ 11(i) $\quad$ conv.

$\because \displaystyle\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)\left(\frac{1}{\cos\theta}\right) = 1$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ $\quad \displaystyle\sum \frac{1}{n}$ $\quad$ 11(i) $\quad$ div.

$\because \displaystyle\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} n^{\frac{1}{n}} = 1$

	a_n	Compare b_n	Thm	Res
(i)	$\sum_{n=1}^{\infty}\sqrt{\frac{\ln n}{n}}$	$\sum \frac{1}{\sqrt{n}}$	10	div.
(j)	$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\ln n}$	$\sum \frac{1}{n}$	11 (iii)	div



$$\longleftrightarrow$$

$\sum n^{-p}$ 大 $\qquad$ $\sum n^{-p}$ 小

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(\frac{n}{\ln n}\right)^{\frac{1}{2}}=\infty$$

$$\sum b_n=\sum\frac{1}{n}=\infty$$

Ratio test and Root test.

Thm (Ratio Test)

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$

ⓐ $0 \le \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

ⓑ $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div

ⓒ $\rho = 1 \Rightarrow$ inconclusive.

Thm $\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

Pf: $-|a_n| \le a_n \le |a_n| \Rightarrow 0 \le a_n + |a_n| \le 2|a_n|$

$\sum |a_n| < \infty \Longrightarrow \sum 2|a_n| < \infty$

$\Rightarrow \sum (a_n + |a_n|) < \infty$

$\therefore \sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ converges

Thm (Root test)

If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

ⓐ $0 \le \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

ⓑ $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div.

ⓒ $\rho = 1 \Rightarrow$ inconclusive

Eg $\begin{cases} \sum \frac{1}{n} = \infty & (\rho = 1) \\ \sum \frac{1}{n^2} < \infty & (\rho = 1) \end{cases}$

(for both ratio test and root test)

Eg (i) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

root test: $\displaystyle\lim_{n\to\infty} \left(\frac{2^n+5}{3^n}\right)^{\frac{1}{n}} = \frac{2}{3}$

$$\left(\frac{2^n}{3^n}\right)^{\frac{1}{n}} < \left(\frac{2^n+5}{3^n}\right)^{\frac{1}{n}} < \left(\frac{5\,2^n}{3^n}\right)^{\frac{1}{n}}$$

Since $\lim 5^{\frac{1}{n}} = 1$

(Sandwich Theorem)

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$ $\quad a_{n+1} = \dfrac{(2(n+1))!}{(n+1)!(n+1)!}$

Ratio: $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)}$

$= 4 \Rightarrow$ divergent.