Examples of divergent Series

Eg I $\left(\lim_{n\to\infty} a_n \neq 0\right)$

$$\sum_{n=1}^{\infty} n^2, \quad \sum_{n=1}^{\infty} (-1)^n, \quad \sum_{n=1}^{\infty} \frac{n}{2n+5}$$

Eg 2: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

<u>Sol</u>: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$

$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$

$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$

$= +\infty \quad \left(\sum_{n=1}^{\infty} a_n \text{ conv.} \not\Leftarrow \lim_{n\to\infty} a_n = 0 \right)$

The Integral Test

Thm: If $a_n > 0$, $a_n = f(n)$

and $f(x)$ is positive

, continuous and decreasing

for all $x > N$, $N \in \mathbb{N}$

Then $\sum_{n=N}^{\infty} a_n < \infty \Leftrightarrow \int_N^{\infty} f(x)dx < \infty$

Rm = Remark

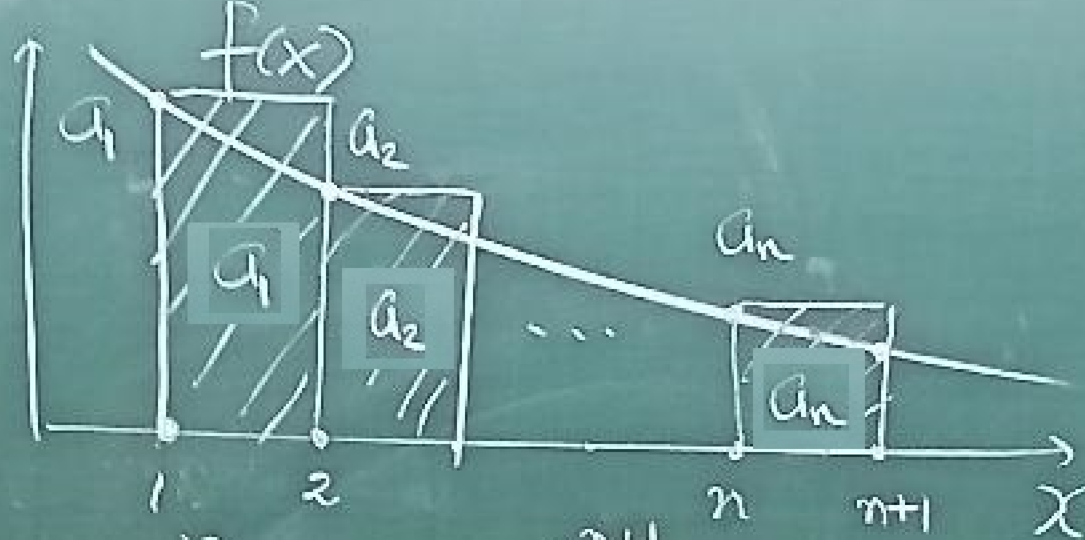
Rm: Since $a_n > 0$

$\sum_{n=N}^{\infty} a_n < \infty \Leftrightarrow \sum_{n=N}^{\infty} a_n$ converges

Eg: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p > 0)$ (Memorize)

Sol: $\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{x^{-p+1}}{-p+1} \Big|_1^{\infty} & < \infty \ (p>1) \\ & = \infty \ (0<p<1) \\ \ln x \Big|_1^{\infty} & = \infty \end{cases}$

Pf: (Take $N=1$ for simplicity)



$$\Rightarrow \sum_{k=1}^{n} a_k > \int_1^{n+1} f(x)\,dx \quad (1)$$

（長方形面積和）

$$\Rightarrow a_1 + \sum_{k=2}^{n} a_k < a_1 + \int_1^{n} f(x)\,dx \quad (2)$$

$$\int_1^{n+1} f(x)\,dx < \sum_{k=1}^{n} a_k < a_1 + \int_1^{n} f(x)\,dx$$

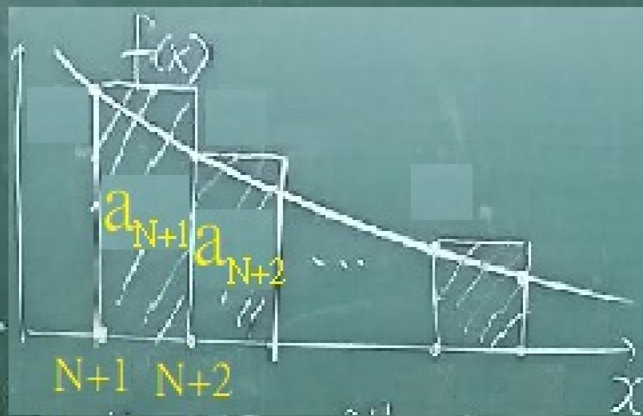
$\therefore$ let $n \to \infty$

$$\int_1^\infty f(x)\,dx = \infty \underset{(1)}{\Rightarrow} \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \infty$$

$$\int_1^\infty f(x)\,dx < \infty \underset{(2)}{\Longrightarrow} \lim_{n\to\infty} \sum_{k=1}^{n} a_k < \infty$$

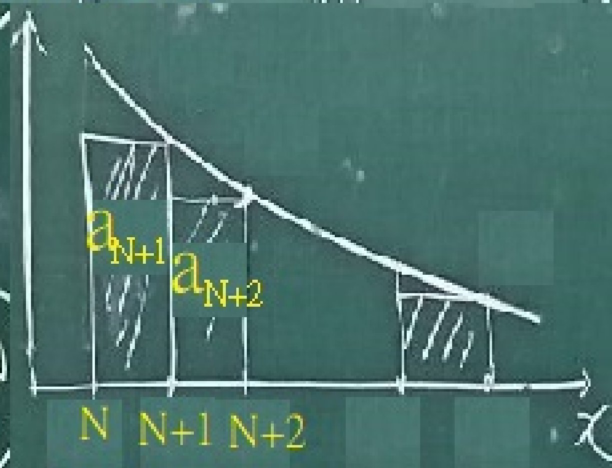
Rm: (Error estimate for convergent series)

let $R_N = a_{N+1} + a_{N+2} + \cdots$
$(= S - S_N)$



$$\int_{N+1}^\infty f(x)\,dx < \sum_{N+1}^{\infty} a_k$$

$(1 \to N+1,\ n \to \infty$ in (1))



$$\sum_{N+1}^{\infty} a_k < \int_N^\infty f(x)\,dx$$

$(1 \to N,\ 2 \to N+1,\ n \to \infty$ in (2))

$$\int_{N+1}^\infty f(x)\,dx < R_N < \int_N^\infty f(x)\,dx$$

Memorize this result. Very important!

Eg3: $\sum_{n=1}^{\infty} n^{-p} \begin{cases} =\infty, & 0<p\leq 1 \\ <\infty, & p>1 \end{cases}$

Eg4: $\sum_{n=0}^{\infty} \frac{1}{1+n^2} < \infty$

Sol:

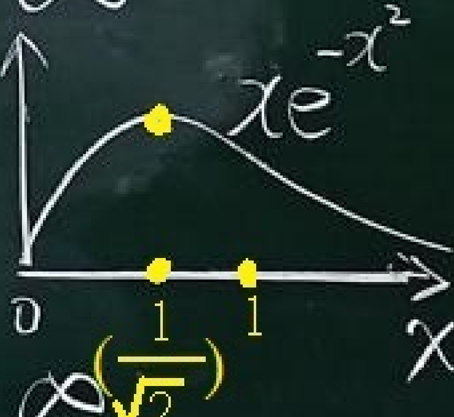
$$\int_0^{\infty} \frac{1}{1+x^2}dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}dx$$

$$= \lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2}$$

Eg5: $\sum_{n=1}^{\infty} n e^{-n^2} < \infty$

Sol: $\int_1^{\infty} x e^{-x^2} dx$

$$= -\frac{1}{2} e^{-x^2} \Big|_1^{\infty} < \infty$$



Eg 6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} = \infty$

Sol: $\displaystyle\lim_{b\to\infty} \int_2^b \frac{1}{2^{\ln x}}\,dx$

$y = \ln x,\ x = e^y$

$dx = e^y\,dy$

$\displaystyle = \lim_{b\to\infty} \int_{y=\ln 2}^{y=\ln b} \frac{1}{2^y} e^y\,dy$

$\displaystyle = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \left(\frac{e}{2}\right)^y dy = \infty$

$\left(\because \frac{e}{2} > 1\right)$

Comparison Test: $\sum_{n=1}^{\infty} a_n$ conv?

If $0 < d_n \leq a_n \leq c_n$

for all $n > N$

Then

① $\sum_{n=1}^{\infty} c_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

② $\sum_{n=1}^{\infty} d_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

Eg 7. $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

Sol $\frac{5}{5n-1} > \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n-1} = \infty$

Rm: $\sum_{n=1}^{\infty} \frac{5}{5n+1}$

Lim

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(1

Sol: $\frac{5}{5n+1} < \frac{1}{n}$

Can not apply comparison test.

Try $\frac{5}{5n+1} > \frac{1}{5n}$

$\left(\frac{5}{5n+1} - \frac{1}{5n} = \frac{20n-5}{5n(5n+1)} > 0 \right)$

$\sum_{n=1}^{\infty} \frac{1}{5n} = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n+1} = \infty$

Limit Comparison Test

If $a_n > 0$, $b_n > 0$, $\forall n \geq N$

(i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$

Then $\sum\limits_{n=1}^{\infty} b_n \begin{matrix} < \infty \\ (= \infty) \end{matrix} \Leftrightarrow \sum\limits_{n=1}^{\infty} a_n \begin{matrix} < \infty \\ (= \infty) \end{matrix}$

(ii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$

Then $\sum\limits_{n=1}^{\infty} b_n < \infty \Rightarrow \sum\limits_{n=1}^{\infty} a_n < \infty$

(iii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$

Then $\sum\limits_{n=1}^{\infty} b_n = \infty \Rightarrow \sum\limits_{n=1}^{\infty} a_n = \infty$

Eg 8 $\sum_{n=1}^{\infty} \frac{5}{5n+1}$

Sol: Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{5}{5n+1}}{\frac{1}{n}} = 1, \quad \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

from (i) $\Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n+1} = \infty$

Eg 9. $\sum_{n=0}^{\infty} \frac{1}{n!} < \infty$

Sol: $1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$

$< 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 2\cdot 2} + \cdots$

$< \infty \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} < \infty$