Recall definition of limit:

$$\lim_{n\to\infty} a_n = \underset{(+\infty)}{L}$$

For every $\underset{(M\in\mathbb{R})}{\varepsilon>0}$,

there exists a corresponding integer N, such that

for all $n > N$,

we have $|a_n - L| < \varepsilon$

$(a_n > M)$

Eg1: $\lim_{n\to\infty} \frac{1}{n} = ?$

"Eg"="Example"

Ans: $L = 0$

$\because$ For any $\varepsilon > 0$

Take $\underline{\underline{N}} = \left[\frac{1}{\varepsilon}\right] + 1 \geq \underline{\underline{\frac{1}{\varepsilon}}}$

"$n > N \Longrightarrow |a_n - 0| < \varepsilon$"

$\because |a_n - 0| = \frac{1}{n} < \frac{1}{N}$

$$\frac{1}{N} \leq \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

#

Eg 2: $\lim_{n\to\infty} n^2 = +\infty$

pf:

For any $M \in \mathbb{R}$

(may assume $M > 0$)

Take $N = [\sqrt{M}] + 1 > \sqrt{M}$

$\Rightarrow$ for all $n > N$,

we have

$$\underset{n^2}{\overset{\|}{a_n}} > \underset{N^2}{\overset{\|}{a_N}} > \sqrt{M}^2 = M$$

Eg3 $\{a_n\} = \{1, -1, 1, -1, \dots\}$

$(a_n = (-1)^{n-1})$

$\Rightarrow \{a_n\}$ diverges

$(\neq \pm\infty)$

Eg4. $\{a_n\} = \{1, 0, 2, 0, 3, 0, \dots\}$

$\lim_{n\to\infty} a_n \neq \begin{matrix} +\infty \\ -\infty \\ \text{any } L \in \mathbb{R} \end{matrix}$

$\{a_n\}$ diverges

How to find $\lim\limits_{n\to\infty} a_n$?

(I): Apply Sandwich Theorem for sequences

(Section 10.1, Theorem 2)

Eg 5: $\lim\limits_{n\to\infty} \dfrac{\cos n}{n}$

Sol: $\dfrac{-1}{n} \leq \dfrac{\cos n}{n} \leq \dfrac{1}{n}$

$\lim\limits_{n\to\infty} \dfrac{-1}{n} = 0 = \lim\limits_{n\to\infty} \dfrac{1}{n}$

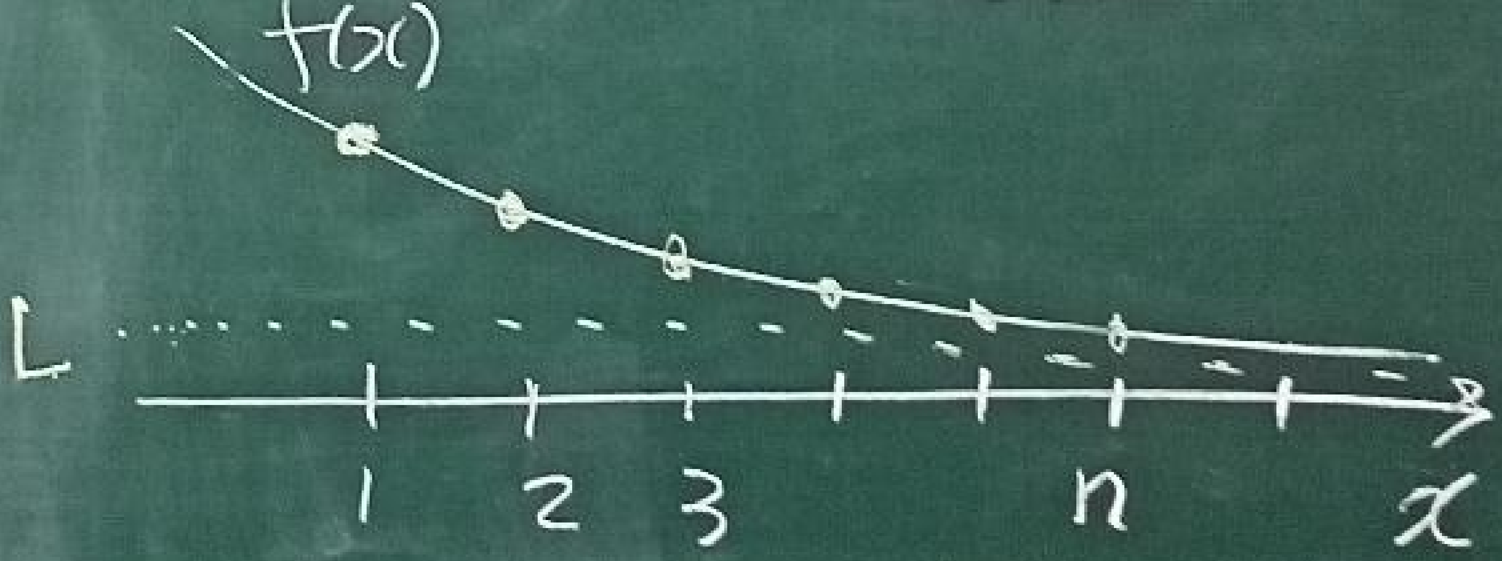
$\xRightarrow{\text{Sandwich Thm}} \lim\limits_{n\to\infty} \dfrac{\cos n}{n} = 0$

(Section 10.1, Theorem 4)

(II): Thm: Suppose $f(x)$ is a function defined for all $x \geq n_0$ with $a_n = f(n)$

and $\lim_{x\to\infty} f(x) = L$ $(\pm\infty)$

then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$



Eg 7: $\lim_{n \to \infty} \frac{\ln n}{n} = ?$

$(Ans = 0)$

Ans: $\frac{``\infty"}{``\infty"}$

find $\lim_{x \to \infty} \frac{\ln x}{x} = ?$

(Here $x \in \mathbb{R}$, $x > 0$).

i.e. $f(x) = \frac{\ln x}{x}$

l'Hopital Rule:

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{(\ln x)'}{x'}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0 \quad \therefore \lim_{n \to \infty} \frac{\ln n}{n} = 0$$

(Momorize this result)

Eg8 $\lim_{n\to\infty} n^{\frac{1}{n}} = ?$ (Ans = 1)

$= \lim_{n\to\infty} \left(e^{\ln n}\right)^{\frac{1}{n}}$

$= \lim_{n\to\infty} e^{\frac{\ln n}{n}}$

(Theorem 3)

$= e^{\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)} = e^{0} = \underset{\sim\sim}{1}$

$\because\ x \xrightarrow{f} e^{x}$

is a continuous function

i.e. $\lim_{x\to x_0} f(x) = f(x_0)$

$= f\left(\lim_{x\to x_0} x\right)$

Memorize this result

Eg 9 $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = ?$ $(\text{Ans}: = e^x)$

$(x\in\mathbb{R}$ is fixed$)$

Ans: "1^∞"

$$= \lim_{n\to\infty}\left(e^{\ln\left(1+\frac{x}{n}\right)}\right)^n$$

$$= \lim_{n\to\infty} e^{\left(\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}\right)}$$

$$= e^{\lim_{n\to\infty}\left(\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}\right)}$$

It remains to find

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{d}{dn}\left(\ln\left(1+\frac{x}{n}\right)\right)}{\frac{d}{dn}\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\cdot\left(\cancel{-}x\,\cancel{n^{-2}}\right)}{\cancel{-}\cancel{n^{-2}}}$$

$$= x$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$$

Eg 10 $\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^{n}$

$= \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^{n}$

$= \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^{n-1}\cdot\left(1+\frac{2}{n-1}\right)$

$= e^{2}\cdot 1 = e^{2}$

Infinite Series

Def: $\sum_{n=1}^{\infty} a_n = L$ $(\pm\infty)$

if $\lim_{n\to\infty} S_n = L$ $(\pm\infty)$

where $S_n = \sum_{k=1}^{n} a_k$

$= (a_1 + a_2 + \cdots + a_n)$

($S_1, S_2, S_3, \ldots, S_n, \ldots$
is the sequence of
partial sum of $\{a_k\}$)

Eg of Convergent Series

* Geometric Series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots$$

$$= \lim_{n\to\infty} S_n, \quad S_n = \sum_{k=0}^{n} ar^k$$

$$= \begin{cases} \dfrac{a}{1-r}, & \text{if } |r|<1 \\ \underline{\text{divergent}}, & \text{if } r = \pm 1 \text{ or } |r|>1 \end{cases}$$

Eg 11: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \text{conv.}$$

* Telescoping Sum

Eg 12: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

<u>Sol</u>: $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$

$= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)$

$+ \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)$

$+ \cancel{\cdots} \cancel{\cdots}$

$+ \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$, $\therefore \lim_{n\to\infty} S_n = 1$

Theorem (10.2: Theorem 7)

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \Longrightarrow \lim_{n\to\infty} a_n = 0$$

$$(\not\Longleftarrow)$$

$$\left(\sum_{n=1}^{\infty} a_n \text{ div.} \Longleftarrow \lim_{n\to\infty} a_n \neq 0 \right)$$

pf: Since $a_n = S_n - S_{n-1}$

$$\therefore \lim_{n\to\infty} a_n = \left(\lim_{n\to\infty} S_n\right) - \left(\lim_{n\to\infty} S_{n-1}\right)$$

$$= L - L = 0$$