Chap 10 Sequences and Series

Why do we care?

Examples:

⊛ $\int_0^1 (1+x^3)^{\frac{1}{2}}\,dx = ?$

⊛ Solve

$$\begin{cases} y''(t) - 2t\,y'(t) - 2y(t) = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

Possible solution method:

⊛ Given $f(x)$, can we find $a_n \in \mathbb{R}$, such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad ?$$

$$\left(x^0 \overset{\text{def}}{=} 1\right)$$

⊛ It is known that, if a_n exists, then

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$\left(0! \overset{\text{def}}{=} 1\right)$$

⊗ Is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

for some $x \neq 0$?

⊗ Is it true that

$$f'(x) \underset{?}{=} \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\int_0^x f(t)\,dt \underset{?}{=} \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Question:

sequence: $\{a_n\}$,
i.e. $(a_0, a_1, \cdots, a_n, \cdots)$

How to define

$$\lim_{n\to\infty} a_n = L \ ?$$

(Recall definition of $\lim_{x\to\infty} f(x) = L$)

Replace x by n
$f(x)$ by a_n

<u>Def</u>: $\lim_{n\to\infty} a_n = L$

if for every $\varepsilon > 0$,

there exists a corresponding integer N

such that

"$n > N \Longrightarrow |a_n - L| < \varepsilon$"

(for all $n > N$, we have $|a_n - L| < \varepsilon$)

Def: $\lim_{n\to\infty} a_n = \begin{matrix} +\infty \\ -\infty \end{matrix}$

if for every $M \in \mathbb{R}$
$(m \in \mathbb{R})$

there exists a corresponding integer N, such that

" $n > N \Longrightarrow a_n > M$ "
$(a_n < m)$

(for all $n > N$,
we have. $a_n > M$
$(a_n < m)$)