Remark on Definition of Differentiability

The following definitions are equivalent (i.e. the same, even though they look different)

Definition 1: $f(x,y)$ is differentiable at (x_0,y_0) if

$\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$ exist and

$$f(x,y) = L(x,y) + \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0) \tag{1}$$

where

$$L(x,y) = f(x_0,y_0) + \partial_x f(x_0,y_0) \cdot (x - x_0) + \partial_y f(x_0,y_0) \cdot (y - y_0) \tag{2}$$

and

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1 = \lim_{(x,y)\to(x_0,y_0)} \varepsilon_2 = 0$$

Definition 2: $f(x,y)$ is differentiable at (x_0,y_0) if

There exists $a,\ b,\ c \in \mathbb{R}$ such that the linear function

$$L(x,y) = a \cdot (x - x_0) + b \cdot (y - y_0) + c$$

satisfies

$$f(x,y) = L(x,y) + \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0) \tag{3}$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1 = \lim_{(x,y)\to(x_0,y_0)} \varepsilon_2 = 0$$

Moreover, if $a,\ b,\ c \in \mathbb{R}$ do exist, we have

$$\partial_x f(x_0,y_0) \text{ and } \partial_y f(x_0,y_0) \text{ exist, and } a = \partial_x f(x_0,y_0),\ b = \partial_y f(x_0,y_0),\ c = f(x_0,y_0). \tag{4}$$

The proof of (4) is identical to the one-variable case. See page 6 of Lecture 14.

Other equivalent definitions:

In view of the identity (see homework 07, problem 2 for hint of proof)

$$\varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0) = \varepsilon \cdot \sqrt{(x - x_0)^2 (y - y_0)^2}, \tag{5}$$

we can further modify definition 1 and definition 2 into following equivalent definitions:

Definition 1': $f(x, y)$ is differentiable at (x_0, y_0) if

$\partial_x f(x_0, y_0)$ and $\partial_y f(x_0, y_0)$ exist and

$$f(x, y) = L(x, y) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2} \tag{6}$$

where

$$L(x, y) = f(x_0, y_0) + \partial_x f(x_0, y_0) \cdot (x - x_0) + \partial_y f(x_0, y_0) \cdot (y - y_0) \tag{7}$$

and

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0 \tag{8}$$

Definition 2': $f(x, y)$ is differentiable at (x_0, y_0) if

There exists $a,\ b,\ c \in \mathbb{R}$ such that the linear function

$$L(x, y) = a \cdot (x - x_0) + b \cdot (y - y_0) + c$$

satisfies

$$f(x, y) = L(x, y) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2} \tag{9}$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0$$

Moreover, if $a,\ b,\ c \in \mathbb{R}$ do exist, we have

$$\partial_x f(x_0, y_0) \text{ and } \partial_y f(x_0, y_0) \text{ exist, and } a = \partial_x f(x_0, y_0),\ b = \partial_y f(x_0, y_0),\ c = f(x_0, y_0). \tag{10}$$

Remark: It is straight forward to generalize to higher dimensional case. For example

Definition 1'-3D: $f(x, y, z)$ is differentiable at (x_0, y_0, z_0) if

$\partial_x f(x_0, y_0, z_0)$, $\partial_y f(x_0, y_0, z_0)$ and $\partial_z f(x_0, y_0, z_0)$ exist and

$$f(x, y, z) = L(x, y, z) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{11}$$

where

$$L(x, y, z) = f(x_0, y_0, z_0) + \partial_x f(x_0, y_0, z_0) \cdot (x - x_0) + \partial_y f(x_0, y_0, z_0) \cdot (y - y_0) + \partial_z f(x_0, y_0, z_0) \cdot (z - z_0) \tag{12}$$

and

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \varepsilon = 0$$

Definitions 1-3D, 2-3D and 2'-3D are similar.

Question: How do we verify whether $f(x, y)$ is differentiable at (x_0, y_0)? (For example, Homework 07 problem 3)

Answer:

Step 1: Find $\partial_x f(x_0, y_0)$ and $\partial_y f(x_0, y_0)$. If they don't exist, then $f(x, y)$ is NOT differentiable at (x_0, y_0).

Step 2: If $\partial_x f(x_0, y_0)$ and $\partial_y f(x_0, y_0)$ exist, we still need to check whether $L(x, y)$ given by (2) satisfies (1) or (6). In most cases, it is easier to check (6). In other words, to check whether the following is true:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0\,? \tag{13}$$

If (13) is true, then we can write

$$\varepsilon \stackrel{\text{def}}{=} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}, \qquad \lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0. \tag{14}$$

which is the same as (6) and (8), thus we conclude that $f(x, y)$ is differentiable at (x_0, y_0) if (13) holds. One can follow the procedures above to solve problem 3 of Homework 07 for $(x_0, y_0) = (0, 0)$.

On the other hand, Definition 1 is more convenient for the proof of Theorem 2 (page 830) given in Appendix 9.

Tangent plane interpretation:

In Definition 1': (6) and (8) leads to

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0, \tag{15}$$

meaning that $f(x, y)$ and $L(x, y)$ are very close, their difference is even much smaller than $\sqrt{(x - x_0)^2 + (y - y_0)^2}$ when $\sqrt{(x - x_0)^2 + (y - y_0)^2} \to 0$. This is the definition for the two surfaces $z = f(x, y)$ and $z = L(x, y)$ to be tangent at $(x_0, y_0, f(x_0, y_0))$.

Linearization (Linear Approximation):

When $f(x, y)$ is differentiable at (x_0, y_0) (and therefore $f(x, y)$ and $L(x, y)$ are very close near (x_0, y_0)), we call $z = L(x, y)$ the linearization or linear approximation (since L is linear) of $z = f(x, y)$ at (x_0, y_0).

Remark on the Definition of tangent plane on page 853:

When $f(x, y)$ is differentiable at (x_0, y_0), the function $F(x, y, z) = z - f(x, y)$ is also differentiable at (x_0, y_0, z_0) for any z_0. In particular, the surface $z = f(x, y)$ passing through $(x_0, y_0, f(x_0, y_0))$ is the same as the level surface $S_0 = \{(x, y, z),\ F(x, y, z) = 0\}$. Therefore, the tangent plane of S_0 at $(x_0, y_0, z_0 = f(x_0, y_0))$ can be obtained from

$$\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0, \qquad z_0 = f(x_0, y_0). \tag{16}$$

It is easy to see that (16) leads to $z = L(x, y)$.