Homework 07

1. Section 14.3: Problems 72.

   Remark: See also page 10 of Lecture 14.

   Hint: Since $(0,0)$ is a special point, we need to compute $f_{xy}(0,0)$ from definition:
   $$f_{xy}(0,0) = \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{y-0}.$$
   For this, we need to compute $f_x(0,y)$ for $y \neq 0$ by direct differentiation and
   $$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0}.$$
   The procedure for computing $f_{yx}(0,0)$ is similar.

2. Section 14.3:

   Show that
   $$h(x,y) = \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0) \text{ with } \lim_{(x,y)\to(x_0,y_0)} (\varepsilon_1, \varepsilon_2) = (0,0) \tag{1}$$
   if and only if
   $$h(x,y) = \varepsilon \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2} \text{ with } \lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0. \tag{2}$$
   That is, if $h(x,y)$ is given by (1), then it can be rewritten (combined) as (2). Also, if $h(x,y)$ is given by (2), then it can be rewritten (split) as (1).

   Remark: If you still remember the 'Big O' and 'small o' notation from last semester, the above statement can be recast as:
   $$o(1) \cdot \Delta x + o(1) \cdot \Delta y = o(1) \cdot \sqrt{(\Delta x)^2 + (\Delta y)^2}, \tag{3}$$
   or
   $$o(\Delta x) + o(\Delta y) = o(\sqrt{(\Delta x)^2 + (\Delta y)^2}), \tag{4}$$
   where the small $o(\cdot)$'s refer to $2D$ limit $\lim_{(\Delta x,\Delta y)\to(0,0)}$, same as $\lim_{(x,y)\to(x_0,y_0)}$.

   The statements (3) or (4) are more concise and widely used. If you forgot the the 'Big O' and 'small o' notation, just stick to the original statements (1) and (2).

   Hint: Use
   $$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\Delta x + \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\Delta y$$
   or
   $$|\Delta x| \le \sqrt{(\Delta x)^2 + (\Delta y)^2}, \quad |\Delta y| \le \sqrt{(\Delta x)^2 + (\Delta y)^2},$$
   $$\sqrt{(\Delta x)^2 + (\Delta y)^2} \le |\Delta x| + |\Delta y|$$

3. Section 14.3: (Optional, try it if time permits)

   Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \tag{5}$$

   (a) Show that

$$\partial_x f(x,y) = \begin{cases} \dfrac{y^3}{(\sqrt{x^2+y^2})^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \tag{6}$$

   Hint: For $(x,y) \neq (0,0)$, compute the derivative directly. For $(x,y) = (0,0)$, find the partial derivative from definition.

   (b) Use Two Path Test to show that $\partial_x f$ is not continuous at $(0,0)$.

   Since $\partial_x f$ is not continuous at $(0,0)$, Theorem 3 is not applicable and therefore the differentiability of f at $(0,0)$ is still inconclusive. To find out whether f is differentiable at $(0,0)$ or not, we continue with the following steps.

   (c) Find the linear function $L(x,y) = f(0,0) + \partial_x f(0,0)(x-0) + \partial_y f(0,0)(y-0)$.

   (d) Check if this $L(x,y)$ satisfies the requirement "(ii)" on page 1 of Lecture 14. In this example, the $\varepsilon \cdot \sqrt{(x-0)^2+(y-0)^2}$ version is easier to check (see page 3 of Lecture 14 and problem 2 above). In the end, you should reach the conclusion that f is not differentiable at $(0,0)$.

4. Section 14.4: Problems 1, 7, 10, 21, 24, 29, 31, 43, 51.

5. Section 14.4: Suppose that $F(x,y,z) = 0$ can implicitly define $x = f(y,z)$, or $y = g(z,x)$, or $z = h(x,y)$ near some point (x_0,y_0,z_0) with $F(x_0,y_0,z_0) = 0$. (for example, $F(x,y,z) = x + 2y + 3z - 4$ can). Show that, for any such point (x_0,y_0,z_0), we have

$$\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}\frac{\partial h}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\frac{\partial h}{\partial y} = -1 \tag{7}$$

   Hint: Read Example 5 and Example 6.

   Remark: Sometimes the identity (7) may be written as

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x = -1 \tag{8}$$

   or simply (and confusingly)

$$\left(\frac{\partial x}{\partial y}\right) \left(\frac{\partial y}{\partial z}\right) \left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial x}{\partial z}\right) \left(\frac{\partial y}{\partial x}\right) \left(\frac{\partial z}{\partial y}\right) = -1 \tag{9}$$

   The identities (8) or (9) are exactly the same as (7) with a somewhat confusing notation.

6. Section 14.5: Problems 9, 15, 19, 25, 27, 29, 35, 36, 40 (See page 850).

7. Section 14.5: Let $f_1(x,y) = \sqrt{x^2+y^2}^{\frac{1}{2}} = (x^2+y^2)^{\frac{1}{4}}$,
$f_2(x,y) = 2x+3y+4+\sqrt{x^2+y^2}^{\frac{3}{2}} = 2x+3y+4+(x^2+y^2)^{\frac{3}{4}}$ and
$f_3(x,y) = \dfrac{x^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$, $f_3(0,0) = 0$.

(a) Are f_i continuous at $(0,0)$?

(b) Do $\partial_x f_i$ and $\partial_y f_i$ exist at $(0,0)$?

(c) Use the definition of directional derivative to evaluate $\left.\dfrac{df_i}{ds}\right|_{(0,0),(\cos\theta,\sin\theta)}$, i.e. the directional derivative of f_i at $(x_0,y_0) = (0,0)$ in the direction $(\cos\theta,\sin\theta)$, if it exists.

(d) Are f_i differentiable at $(0,0)$?